Example on Geometric Series

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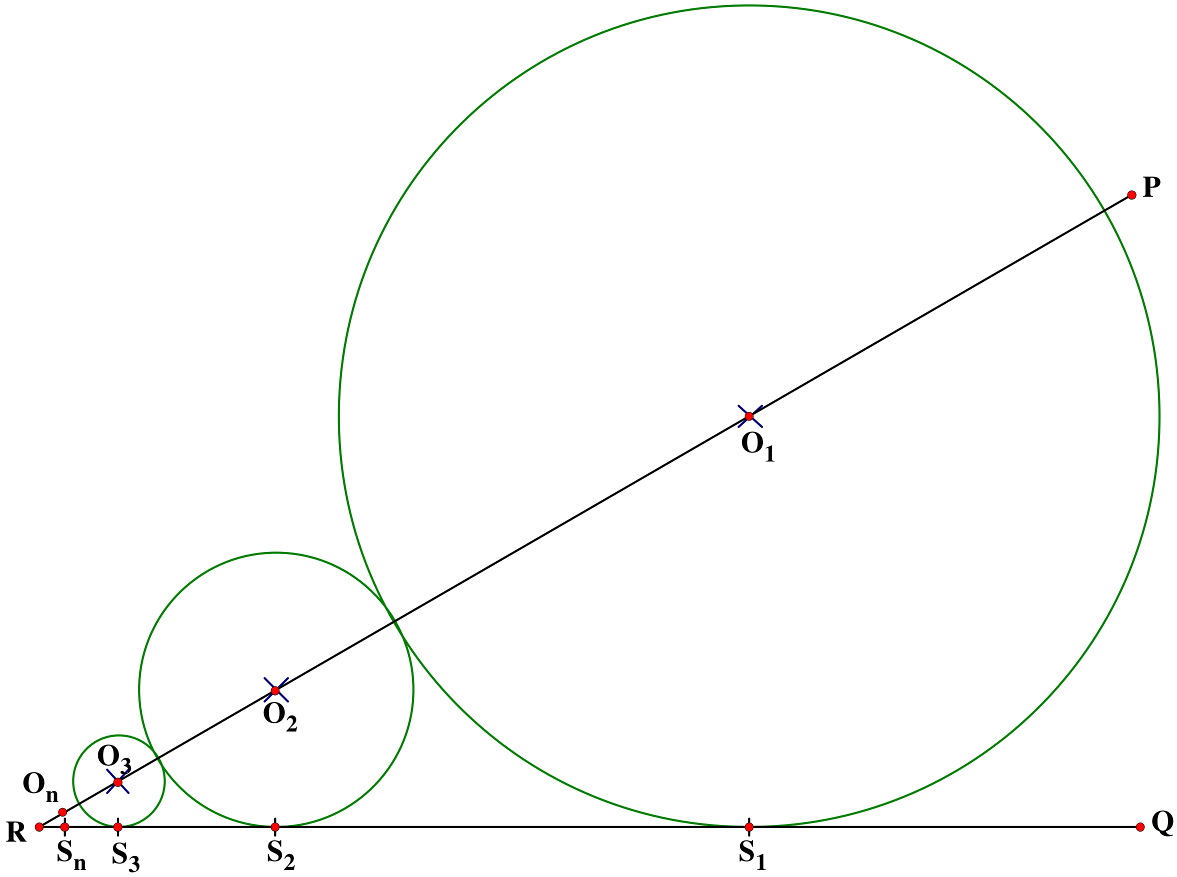


在圖中，兩條直綫 PR 和 QR 相交於 R 點。PR 上的點 O_1、O_2、O_3、…、O_n 都是一些圓的圓心，使

(I) $O_1R > O_2R > O_3R > \cdots > O_nR$；

(II) 以 O_1 為圓心的圓與 RQ 相切於 S_1，以 O_2 為圓心的圓與 RQ 相切於 S_2，以 O_3 為圓心的圓與 RQ 相切於 S_3，餘此類推；

(III) 以 O_1 為圓心的圓與以 O_2 為圓心的圓互相外切，而以 O_2 為圓心的圓與以 O_3 為圓心的圓互相外切，餘此類推。

如果 r_1、r_2、…、r_n 分別表示以 O_1、O_2、…、O_n 為圓心的圓的半徑，而 $r_1 = 10$，$O_1R = 20$。

(a) (i) 試以 r_2 表示 O_1O_2 和 O_2R，

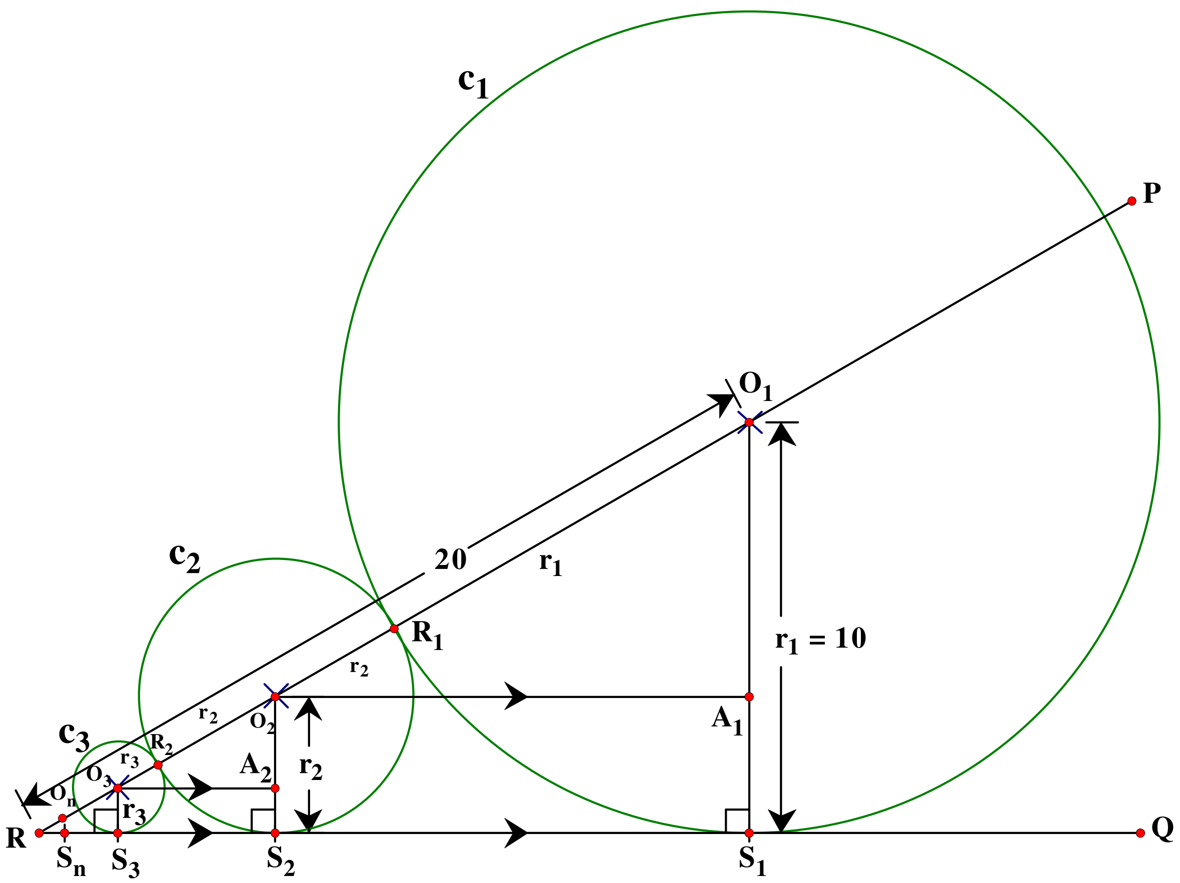
(ii) 從而，求 r_2；

(b) (i) 試以 r_3 表示 O_3R，

(ii) 從而，求 r_3；

(c) 試求 r_5；

(d) 如果這個作圖的過程一直繼續下去，試求所有圓的面積的總和。



(a) (i) Label the largest circle as c_1, the second largest circle as c_2 , the third larest circle as c_3, $\cdots$ and so on.

c_1 touches c_2 at R_1 , c_2 touches c_3 at R_2 , $\cdots$ and so on.

$O_1S_1 \perp QR$, $O_2S_2 \perp QR$, $O_3S_3 \perp QR$, $\cdots$ and so on (tangent $\perp$ radii)

$O_1S_1 = O_1R_1 = r_1$, $O_2S_2 = O_2R_2 = r_2$, $O_3S_3 = r_3$, $\cdots$ and so on.

$O_1O_2 = r_1 + r_2 = 10 + r_2$

$\Delta O_2RS_2 \sim \Delta O_1RS_1$ (A.A.A.)

$\frac{O_2R}{O_1R} = \frac{O_2S_2}{O_1S_1}$ (cor. sides, ~Δs)

$O_2R = 20 \times \frac{r_2}{10} = 2r_2$

(ii) Draw O_2A_1 // S_2S_1, cutting O_1S_1 at A_1.

Then $\angle O_1A_1O_2 = \angle O_1S_1S_2 = 90°$ (cor. ∠s O_2A_1 // S_2S_1)

$A_1O_2S_2S_1$ is a rectangle (It has 3 right angles)

$A_1S_1 = O_2S_2 = r_2$ (opp. sides of rectangle)

$O_1A_1 = O_1S_1 - A_1S_1 = r_1 - r_2 = 10 - r_2$

$\Delta O_1O_2A_1 \sim \Delta O_1RS_1$ (A.A.A.)

$\frac{O_1A_1}{O_1O_2} = \frac{O_1S_1}{O_1R}$ (cor. sides, ~Δs)

$\frac{10 - r_2}{10 + r_2} = \frac{10}{20} = \frac{1}{2}$

$20 - 2r_2 = 10 + r_2$

$r_2 = \frac{10}{3}$

(b) (i) $\Delta O_3RS_3 \sim \Delta O_1RS_1$ (A.A.A.)

$\dfrac{O_3R}{O_1R} = \dfrac{O_3S_3}{O_1S_1}$ (cor. sides, ~Δs)

$O_3R = 20 \times \dfrac{r_3}{10} = 2r_3$

(ii) Draw $O_3A_2 \parallel S_3S_2$, cutting O_2S_2 at A_2 .

Then $\angle O_2A_2O_3 = \angle O_2S_2S_3 = 90°$ (cor. ∠s $O_2A_1 \parallel S_2S_1$)

$A_2O_3S_3S_2$ is a rectangle (It has 3 right angles)

$A_2S_2 = O_3S_3 = r_3$ (opp. sides of rectangle)

$O_2A_2 = O_2S_2 - A_2S_2 = r_2 - r_3$

$\Delta O_2O_3A_2 \sim \Delta O_1RS_1$ (A.A.A.)

$\dfrac{O_2A_2}{O_2O_3} = \dfrac{O_1S_1}{O_1R}$ (cor. sides, ~Δs)

$\dfrac{r_2 - r_3}{r_2 + r_3} = \dfrac{10}{20} = \dfrac{1}{2}$

$2r_2 - 2r_3 = r_2 + r_3$

$r_3 = \dfrac{1}{3}r_2 = \dfrac{1}{3}\left(\dfrac{10}{3}\right) = \dfrac{10}{9}$

(c) $r_1 = 10$, $r_2 = \dfrac{10}{3}$, $r_3 = \dfrac{10}{9}$, form a geometric sequence with common ratio $\dfrac{1}{3}$

$r_5 = 10 \times \left(\dfrac{1}{3}\right)^{5-1} = \dfrac{10}{81}$

(d) Sum of areas of all cirles = $\pi(10)^2 + \pi\left(\dfrac{10}{3}\right)^2 + \pi\left(\dfrac{10}{9}\right)^2 + \cdots$ to infinity

The above series is a geometric series sum to infinity with common ratio $= \dfrac{1}{9}$

$S_\infty = \dfrac{a}{1-R} = \dfrac{100\pi}{1-\frac{1}{9}} = \dfrac{225\pi}{2}$