Series Example (HKAL 1992 Paper 1 Q10)

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Let $\{a_1, a_2, ...\}$, $\{b_1, b_2, ...\}$ be two sequence of real numbers, and $b_0 = 0$.

(*a*) Show that $\sum_{i=1}^{k} a_i(b_i - b_{i-1}) = a_k b_k + \sum_{i=1}^{k-1}(a_i - a_{i+1})b_i$, $k = 2, 3, \ldots$.

(*b*) Suppose $\{a_i\}$ is decreasing and $|b_i| \le K$ for all i, where K is a constant.

Show that $\left|\sum_{i=1}^{k} a_i(b_i - b_{i-1})\right| \le K\{|a_1| + 2|a_k|\}$, $k = 1, 2, \ldots$.

(*c*) Using (*b*), or otherwise, show that for any positive integers n and p,

$$\left|\sum_{i=n}^{n+p} \frac{(-1)^i}{i}\right| \le \frac{3}{2n}$$

(**Remark**: the marking scheme in part (*c*) is wrong, thank for Mr. Ng Ka Lok in Wah Yan College, Kowloon for pointing out this.)

(*a*)

$$\sum_{i=1}^{k} a_i(b_i - b_{i-1}) = a_1(b_1 - b_0) + a_2(b_2 - b_1) + \cdots + a_k(b_k - b_{k-1})$$

$$= a_1b_1 + a_2b_2 - a_2b_1 + \cdots + a_kb_k - a_kb_{k-1}, \because b_0 = 0$$

$$= (a_1 - a_2)b_1 + (a_2 - a_3)b_2 + \cdots + (a_{k-1} - a_k)b_{k-1} + a_kb_k$$

$$= a_kb_k + \sum_{i=1}^{k-1}(a_i - a_{i+1})b_i$$

(*b*) Given $a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n \ge \ldots\ldots$ and $|b_i| \le K$ for $i = 0, 1, 2, \ldots$

$$\left|\sum_{i=1}^{k} a_i(b_i - b_{i-1})\right| = \left|a_kb_k + \sum_{i=1}^{k-1}(a_i - a_{i+1})b_i\right|$$

$$\le |a_kb_k| + \left|\sum_{i=1}^{k-1}(a_i - a_{i+1})b_i\right|$$

$$\le K\,|a_k| + \sum_{i=1}^{k-1}|(a_i - a_{i+1})||bi|$$

$$\le K\,|a_k| + K\sum_{i=1}^{k-1}|(a_i - a_{i+1})|$$

$$\le K\left\{|a_k| + \sum_{i=1}^{k-1}(a_i - a_{i+1})\right\}, \because a_i - a_{i+1} \ge 0$$

$$\le K\{|a_k| + a_1 - a_k\}$$

$$\le K\{|a_1| + 2|a_k|\}$$

(*c*) Case 1 $p = 2m + 1$,

$$\sum_{i=n}^{n+p}\frac{(-1)^i}{i} = \left[\frac{(-1)^n}{n} + \frac{(-1)^{n+1}}{n+1} + \cdots + \frac{(-1)^{n+p}}{n+p}\right]$$

$$= (-1)^n\left[\frac{1}{n} - \frac{1}{n+1} + \cdots + \frac{(-1)^{2m+1}}{n+2m+1}\right]$$

$$= (-1)^n\left[\left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \cdots + \left(\frac{1}{n+2m} - \frac{1}{n+2m+1}\right)\right]$$

On the other hand,

$$\sum_{i=n}^{n+p}\frac{(-1)^i}{i}=(-1)^n\left[\frac{1}{n}-\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-\left(\frac{1}{n+3}-\frac{1}{n+4}\right)-\cdots-\left(\frac{1}{n+2m-1}-\frac{1}{n+2m}\right)-\frac{1}{n+2m+1}\right]$$

$\because 0<\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)+\cdots+\left(\frac{1}{n+2m}-\frac{1}{n+2m+1}\right)$ and

$$\frac{1}{n}-\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-\left(\frac{1}{n+3}-\frac{1}{n+4}\right)-\cdots-\left(\frac{1}{n+2m-1}-\frac{1}{n+2m}\right)-\frac{1}{n+2m+1}<\frac{1}{n}$$

$$\therefore 0<\left|\sum_{i=n}^{n+p}\frac{(-1)^i}{i}\right|<\frac{1}{n}=\frac{2}{2n}<\frac{3}{2n}$$

Case 2 $p = 2m$

$$\sum_{i=n}^{n+p}\frac{(-1)^i}{i}=\left[\frac{(-1)^n}{n}+\frac{(-1)^{n+1}}{n+1}+\cdots+\frac{(-1)^{n+p}}{n+p}\right]$$

$$=(-1)^n\left[\frac{1}{n}-\frac{1}{n+1}+\cdots+\frac{(-1)^{2m}}{n+2m}\right]$$

$$=(-1)^n\left[\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)+\cdots+\left(\frac{1}{n+2m-2}-\frac{1}{n+2m-1}\right)+\frac{1}{n+2m}\right]$$

On the other hand,

$$\sum_{i=n}^{n+p}\frac{(-1)^i}{i}=(-1)^n\left[\frac{1}{n}-\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-\left(\frac{1}{n+3}-\frac{1}{n+4}\right)-\cdots-\left(\frac{1}{n+2m-1}-\frac{1}{n+2m}\right)\right]$$

$\because 0<\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)+\cdots+\left(\frac{1}{n+2m-2}-\frac{1}{n+2m-1}\right)+\frac{1}{n+2m}$ and

$$\frac{1}{n}-\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-\left(\frac{1}{n+3}-\frac{1}{n+4}\right)-\cdots-\left(\frac{1}{n+2m-1}-\frac{1}{n+2m}\right)<\frac{1}{n}$$

$$\therefore 0<\left|\sum_{i=n}^{n+p}\frac{(-1)^i}{i}\right|<\frac{1}{n}=\frac{2}{2n}<\frac{3}{2n}$$

The proof is completed.

In fact, a smaller upper bound for $\left|\sum_{i=n}^{n+p}\frac{(-1)^i}{i}\right|$ is $\frac{1}{n}$.