Limit of Sequence Example

Created by Mr. Francis Hung Last updated: February 10, 2022

The following notes try to show that for $0 < p < 1$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Step 1: $\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=\infty$

First, we prove by mathematical induction that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^m} \ge 1+\frac{m}{2} \quad \forall m \in \mathbf{N}$$

$m = 1$, LHS $=1+\frac{1}{2}$; RHS $=1+\frac{1}{2}$. $\therefore$LHS $\ge$ RHS

It is true for $m = 1$.

Suppose that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k} \ge 1+\frac{k}{2}$ for some $k \in \mathbf{N}$,

$m = k + 1$, LHS $=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k}+\underbrace{\frac{1}{2^k+1}+\cdots+\frac{1}{2^{k+1}}}_{2^k \text{ terms}}$

$$\ge 1+\frac{k}{2}+\underbrace{\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}+\cdots+\frac{1}{2^{k+1}}}_{2^k \text{ equal terms}}$$

$$\ge 1+\frac{k}{2}+\frac{2^k}{2^{k+1}}$$

$$=1+\frac{k}{2}+\frac{1}{2}$$

$$=1+\frac{k+1}{2}=\text{RHS}$$

It is also true for $m = k + 1$ if it is true for $m = k$.

By the principle of mathematical induction, $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^m} \ge 1+\frac{m}{2} \quad \forall m \in \mathbf{N}$

Now for $n > 1$, $\exists m \in \mathbf{N}$ such that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \ge 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^m} \ge 1+\frac{m}{2}$

$\because \lim_{m\to\infty}\left(1+\frac{m}{2}\right)=\infty$

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=\infty$

Step 2: $\lim_{n\to\infty}\log n=\infty$

Proof: Consider the curve $y=\frac{1}{x}$ for $1 \le x \le n$.

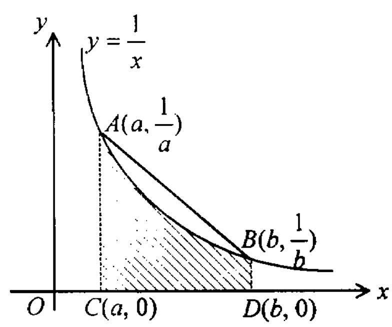
Area under the curve $\ge$ sum of areas of rectangles width = 1 under the curve

$\int_1^n \frac{1}{x}\mathrm{d}x \ge \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$

$\ln x\Big|_1^n \ge \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$

$\ln n \ge \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$

$\therefore \lim_{n\to\infty}\log n \ge \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-1=\infty$



Step 3: For $0 < p < 1$, $1 \le x \le n$

$x^p \ge 1$

$$\frac{x^p}{x} \ge \frac{1}{x}$$

$$\frac{1}{x^{1-p}} \ge \frac{1}{x} \text{ for } 1 \le x \le n$$

$$\int_1^n \frac{1}{x^{1-p}}\,\mathrm{d}x \ge \int_1^n \frac{1}{x}\mathrm{d}x$$

$$\left.\frac{x^p}{p}\right|_1^n \ge \ln x\Big|_1^n$$

$$\frac{n^p - 1}{p} \ge \ln n$$

$n^p \ge p \ln n + 1$

$$\because \lim_{n\to\infty}\left(p\ln n + 1\right) = \infty$$

$$\therefore \lim_{n\to\infty} n^p = \infty$$

$$\therefore \lim_{n\to\infty}\frac{1}{n^p} = \frac{1}{\infty} = 0$$