Limit of sequence lecture notes

Reference: Limit and continuity by C.S. Lee 1986 Fillans Limited

Edited by Mr. Francis Hung on March 28, 2009 Last updated: 24 March 2023

1. Definition of limit of sequence.

Consider the sequence: $\{x_n\}_{n=1}^{\infty}$.

$\lim_{n\to\infty} x_n$ exists and equal to ℓ if the following condition is satisfied:

$\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall\ n > N$, $|x_n - \ell| < \varepsilon$.

Otherwise, we say that $\lim_{n\to\infty} x_n$ does not exist; or the sequence diverges or not convergent.

Example 1.1

Consider the sequence: $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

$\forall\ \varepsilon > 0$, let $N = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{N}$, then $N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{N} < \varepsilon$

such that $\forall\ n > N$, $|x_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N} < \varepsilon$.

$\therefore\ \lim_{n\to\infty} \frac{1}{n} = 0$

Example 1.2

Consider the sequence: $\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$.

$\forall\ \varepsilon > 0$, let $N = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{N}$, then $N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{N} < \varepsilon$

such that $\forall\ n > N$, $|x_n - 1| = \left|\frac{n-1}{n} - 1\right| = \frac{1}{n} \le \frac{1}{N} < \varepsilon$.

$\therefore\ \lim_{n\to\infty} \frac{n-1}{n} = 1$

Example 1.3

Consider the sequence: $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$.

$\forall\ \varepsilon > 0$, let $N = \left|\left[\frac{\log\frac{1}{\varepsilon}}{\log 2}\right]\right| + 1 \in \mathbb{N}$, then $N > \frac{\log\frac{1}{\varepsilon}}{\log 2} \Rightarrow N \log 2 > \log\frac{1}{\varepsilon} \Rightarrow 2^N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{2^N} < \varepsilon$

such that $\forall\ n > N$, $|x_n - 0| = \left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} \le \frac{1}{2^N} < \varepsilon$.

$\therefore\ \lim_{n\to\infty} \frac{1}{2^n} = 0$

Example 1.4

Consider the sequence: $\{\alpha^n\}_{n=1}^{\infty}$, where $0 \le \alpha$.

Claim: Bernoulli inequality If $x \ge -1$, then $(1 + x)^n \ge 1 + nx$, $\forall n \in \mathbb{N}$

Proof: Induction on n. $n = 1$, $(1 + x)^1 = 1 + x$, the result is obvious.

Suppose $(1 + x)^k \ge 1 + kx$

Multiply both sides by $(1 + x)$, which is non-negative.

$(1 + x)^{n+1} \ge (1 + nx)(1 + x)$

$(1 + x)^{n+1} \ge 1 + (n + 1)x + nx^2 \ge 1 + (n + 1)x$

By MI, if $x \ge -1$, then $(1 + x)^n \ge 1 + nx$, $\forall n \in \mathbb{N}$

If $\alpha = 0$, $\lim_{n\to\infty} \alpha^n = 0$; if $\alpha = 1$, $\lim_{n\to\infty} \alpha^n = 1$

If $\alpha > 1$, let $\alpha = 1 + x$; where $x > 0$

By Bernoulli inequality, $(1 + x)^n \ge 1 + nx \Rightarrow (1 + x)^n \ge nx$ ⋯⋯ (1)

Claim $\lim_{n\to\infty} \alpha^n$ does not exist for $\alpha > 1$.

Proof: Suppose on the contrary, $\lim_{n\to\infty} \alpha^n$ exists and equal to ℓ.

Clearly $\alpha^n > 0$ and $\ell > 0$

$\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall\ n > N$, $|\alpha^n - \ell| < \varepsilon$

$-\varepsilon < \alpha^n - \ell < \varepsilon$

$-\varepsilon + \ell < \alpha^n < \varepsilon + \ell$

$nx \le (1 + x)^n < \varepsilon + \ell$

$n \le \frac{\varepsilon + \ell}{x}$

Let $\varepsilon = \ell$, $n \le \frac{2\ell}{x}$ $\forall\ n > N$, which means that n is bounded above by $\frac{2\ell}{x}$.

That is a contradiction.

$\therefore$ $\lim_{n\to\infty} \alpha^n$ does not exist for $\alpha > 1$.

If $0 < \alpha < 1$, let $\alpha = \frac{1}{1+x}$; where $x > 0$

By Bernoulli inequality, $(1 + x)^n \ge 1 + nx \Rightarrow \frac{1}{(1+x)^n} \le \frac{1}{nx}$ ⋯⋯ (2)

$\forall\ \varepsilon > 0$, let $N = \left[\frac{1}{x\varepsilon}\right] + 1 \in \mathbb{N}$, then $N \ge \frac{1}{x\varepsilon} \Rightarrow \frac{1}{Nx} \le \varepsilon$

$\forall\ n > N$, $n > \frac{1}{x\varepsilon}$ $|\alpha^n - 0| = \alpha^n = \frac{1}{(1+x)^n} \le \frac{1}{nx} < \varepsilon$

$\therefore$ $\lim_{n\to\infty} \alpha^n = 0$

Example 1.5

Prove that if $a > 1$, then $\lim_{n\to\infty}\sqrt[n]{a} = 1$.

Let $\sqrt[n]{a} = 1 + \alpha_n$, $\alpha_n > 0$

By Bernoulli inequality, $a = (1 + \alpha_n)^n \geq 1 + n\alpha_n$

$\alpha_n \leq \frac{a-1}{n}$

Hence, for any given $\varepsilon > 0$, let $N = \left[\frac{a-1}{\varepsilon}\right] + 1 \Rightarrow N > \frac{a-1}{\varepsilon} \Rightarrow \varepsilon > \frac{a-1}{N}$

$\forall\ n > N$, $|\sqrt[n]{a} - 1| = |1 + \alpha_n - 1| = |\alpha_n| = \alpha_n \leq \frac{a-1}{n} < \frac{a-1}{N} < \varepsilon$

Thus, by definition, $\lim_{n\to\infty}\sqrt[n]{a} = 1$.

Example 1.6

Prove that if $0 < \alpha < 1$, then $\lim_{n\to\infty} n\alpha^n = 0$.

$\frac{1}{\alpha} > 1$; Let $\frac{1}{\alpha} = 1 + h$, where $h > 0$.

$\frac{1}{\alpha^n} = (1 + h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \cdots > \frac{n(n-1)}{2}h^2$ for $n \geq 2$

$\alpha^n < \frac{2}{n(n-1)h^2} \Rightarrow n\alpha^n < \frac{2}{(n-1)h^2}$

For any given $\varepsilon > 0$, let $N = \left[\frac{2}{\varepsilon h^2}\right] + 2 \Rightarrow N - 1 > \frac{2}{\varepsilon h^2} \Rightarrow \varepsilon > \frac{2}{(N-1)h^2}$

$\forall\ n > N$, $|n\alpha^n - 0| = n\alpha^n < \frac{2}{(n-1)h^2} < \frac{2}{(N-1)h^2} < \varepsilon$

Thus, by definition, $\lim_{n\to\infty} n\alpha^n = 0$.

2. Divergent sequences

(i) The sequence $a_n = 3^{2n-1}$ diverges to positive infinity.

(ii) The sequence $b_n = 1 - 2n$ diverges to negative infinity.

(iii) The sequence $c_n = (-1)^n$ oscillates (between 1 and –1).

(iv) The sequence $d_n = (-1)^n \cdot n$ oscillates divergent (to $\pm\infty$).

Definition 2.1: The sequence $\{a_n\}$ is said to tend to infinity ($+\infty$) if given any real number M (however large), there exists $N \in \mathbb{N}$ such that $a_n > M$ for all $n > N$.

We write $\lim_{n\to\infty} a_n = \infty$.

Similarly, we write $\lim_{n\to\infty} a_n = -\infty$ if given any real number M (however small), there exists $N \in \mathbb{N}$ such that $a_n < M$ for all $n > N$.

It should be emphasized that ∞ and $-\infty$ are not positive numbers and the sequences are not convergent. Thus,

(i) $\lim_{n\to\infty} 3^{2n-1} = \infty$

(ii) $\lim_{n\to\infty}(1-2n) = -\infty$.

Example 2.1

Prove by definition that (a) $\lim_{n\to\infty} 3^{2n-1} = \infty$; (b) $\lim_{n\to\infty}(1-2n) = -\infty$.

(a) $\forall M \in \mathbb{R}$, let $N = \left[\frac{1}{2}\left(\frac{\log|M|}{\log 3} + 1\right)\right] + 1$, then $3^{2N-1} > M$

$\forall n > N$, $3^{2n-1} > 3^{2N-1} > M$

$\therefore \lim_{n\to\infty} 3^{2n-1} = \infty$

(b) $\forall M \in \mathbb{R}$, let $N = \left|\left[\frac{1-M}{2}\right]\right| + 1$, then $1 - 2N < M$

$\forall n > N$, $1 - 2n < 1 - 2N < M$

$\therefore \lim_{n\to\infty}(1-2n) = -\infty$

2. Divergent sequence

Example 2.2

Let $a_n = \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}}\right)$. (sum of n + 1 terms) Prove that $\lim_{n\to\infty} a_n = \infty$.

Observe that the smallest term is $\frac{1}{\sqrt{2n}}$.

$\forall\ M \in \mathbb{R}$, let $N = [2M^2] + 1$, then $N > 2M^2 \Rightarrow \sqrt{\frac{N}{2}} > M$

$$\forall\ n > N,\ \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}} > \underbrace{\frac{1}{\sqrt{2n}} + \cdots + \frac{1}{\sqrt{2n}}}_{n+1 \text{ terms}} = \frac{n+1}{\sqrt{2n}} > \frac{n}{\sqrt{2n}} = \sqrt{\frac{n}{2}} > \sqrt{\frac{N}{2}} > M$$

$\therefore a_n > M \Rightarrow \lim_{n\to\infty} a_n = \infty$

Definition 2.2

If a_n does not tend to a limit or to ∞ or to –∞, we say that a_n oscillates (or is an oscillating sequence). If a_n oscillates and is bounded, it **oscillates finitely**. If a_n oscillates and is not bounded, it **oscillates infinitely**.

Example 2.3

1. $c_n = (-1)^n$ oscillates finitely (between 1 and –1).

2. $d_n = (-1)^n \cdot n$ oscillates infinitely.

3. The sequence $a_n = \frac{(-1)^n}{n}$ is not an oscillating sequence. It has a limit = 0.

3. Uniqueness of limit

Theorem 3.1 A sequence can converge to only one limit, i.e. **if a limit exists, it is unique**.

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be the given sequence. Try show that if $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$, then $a = b$.

By hypothesis, given any $\varepsilon > 0$, we can find N_1 and $N_2 \in \mathbb{N}$ such that

$|x_n - a| < \frac{\varepsilon}{2}$ whenever $n > N_1$

and $|x_n - b| < \frac{\varepsilon}{2}$ whenever $n > N_2$

then, whenever $n > N = \max\{N_1, N_2\}$, we have

$|a - b| = |a - x_n + x_n - b| \le |a - x_n| + |x_n - b|$ (by triangle inequality)

$= |x_n - a| + |x_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

i.e. $|a - b|$ is less than any positive ε (however small) and so must be zero. Thus $a = b$.

Example 3.1 If $a_n = \sin\frac{1}{n}$, then $|a_n| = \left|\sin\frac{1}{n}\right| \le 1$, therefore, a_n is a bounded sequence.

Theorem 3.2 **If $\lim_{n\to\infty} x_n = \ell$, then the sequence is bounded**.

Proof: $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall\ n > N$, $|x_n - \ell| < \varepsilon$.

Let $\varepsilon = 1$, $\big|\,|x_n| - |\ell|\,\big| < |x_n - \ell| < 1$

$-1 < |x_n| - |\ell| < 1$

$|x_n| < 1 + |\ell|\ \forall\ n > N$

Let $M = \text{Max}\{|x_1|, |x_2|, \cdots, |x_N|, 1 + |\ell|\}$

$\therefore\ \{x_n\}_{n=1}^{\infty}$ is bounded above by M.

We remark that

(1) In other words, if $\{a_n\}$ is bounded, $\{a_n\}$ is not convergent. For example, the sequence $\{a_n\}$ defined by $a_n = \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}}\right)$ is not convergent. Since

$a_n = \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}} > \frac{n}{\sqrt{2n}} = \sqrt{\frac{n}{2}} > \sqrt{\frac{N}{2}} > M$ (however large)

this mean $\{a_n\}$ is not bound, therefore $\{a_n\}$ is not convergent.

(2) The converse of the theorem is not true in general. For example, the sequence $\{a_n\}$ defined by $a_n = (-1)^n$, then $-1 \le a_n \le 1$, that is, $\{a_n\}$ is bounded, but $\{a_n\}$ is not convergent.
It is an oscillating sequence.

4. Theorems on limits

Theorem 4.1 If $\lim_{n\to\infty} a_n = \ell_1$ and $\lim_{n\to\infty} b_n = \ell_2$, then (1) $\lim_{n\to\infty}(a_n + b_n) = \ell_1 + \ell_2$; (2) $\lim_{n\to\infty}(a_n - b_n) = \ell_1 - \ell_2$.

Proof: By hypothesis, for any given $\varepsilon > 0$, we can find N_1 and N_2 such that

$|a_n - \ell_1| < \frac{1}{2}\varepsilon$ for all $n > N_1$, and

$|b_n - \ell_2| < \frac{1}{2}\varepsilon$ for all $n > N_2$

then, for any given $\varepsilon > 0$, we can find $N = \max\{N_1, N_2\}$ such that

$$\begin{aligned}|(a_n + b_n) - (\ell_1 + \ell_2)| &= |(a_n - \ell_1) + (b_n - \ell_2)| \\ &\le |a_n - \ell_1| + |b_n - \ell_2| \quad \text{by triangle inequality} \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \text{ for all } n > N\end{aligned}$$

By definition, we have $\lim_{n\to\infty}(a_n + b_n) = \ell_1 + \ell_2$.

Also,
$$\begin{aligned}(a_n - b_n) - (\ell_1 - \ell_2)| &= |(a_n - \ell_1) + (\ell_2 - b_n)| \\ &\le |a_n - \ell_1| + |\ell_2 - b_n| \quad \text{by triangle inequality} \\ &= |a_n - \ell_1| + |b_n - \ell_2| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \text{ for all } n > N\end{aligned}$$

By definition, we have $\lim_{n\to\infty}(a_n - b_n) = \ell_1 - \ell_2$.

Theorem 4.2 If $\lim_{n\to\infty} a_n = \ell_1$ and $\lim_{n\to\infty} b_n = \ell_2$, then $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = \ell_1\ell_2$

Proof: Since $\lim_{n\to\infty} a_n = \ell_1$, it is bounded by P, i.e. $|a_n| \le P$ for all $n \in \mathbb{N}$ for some positive constant P.

By hypothesis, for any given $\varepsilon > 0$, we can find N_1 and N_2, such that

$|a_n - \ell_1| < \frac{\varepsilon}{2(|\ell_2|+1)}$ for all $n > N_1$, and

$|b_n - \ell_2| < \frac{\varepsilon}{2P}$ for all $n > N_2$

Now, for any given $\varepsilon > 0$, we can find $N = \max\{N_1, N_2\}$ such that

$$\begin{aligned}|a_n b_n - \ell_1\ell_2| &= |a_n(b_n - \ell_2) + \ell_2(a_n - \ell_1)| \\ &\le |a_n||b_n - \ell_2| + |\ell_2||a_n - \ell_1| \quad \text{by triangle inequality} \\ &\le P|b_n - \ell_2| + (|\ell_2| + 1)|a_n - \ell_1| \\ &< P\cdot\frac{\varepsilon}{2P} + (|\ell_2|+1)\cdot\frac{\varepsilon}{2(|\ell_2|+1)} \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \text{ for all } n > N\end{aligned}$$

Therefore, by definition, we have $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = \ell_1\ell_2$.

Lemma If $\lim_{n\to\infty} b_n = \ell_2 \ne 0$, then there exists a natural number N such that $|b_n| > \frac{1}{2}|\ell_2|$ for all $n > N$.

Proof: By hypothesis we can find N such that $|b_n - \ell_2| < \frac{1}{2}|\ell_2|$ for all $n > N$

$$\begin{aligned}|\ell_2| &= |\ell_2 - b_n + b_n| \le |\ell_2 - b_n| + |b_n| \\ &= |b_n - \ell_2| + |b_n| < \frac{1}{2}|\ell_2| + |b_n| \text{ for all } n > N.\end{aligned}$$

Which gives $|b_n| > \frac{1}{2}|\ell_2|$ for all $n > N$.

4. Theorems on limits

Theorem 4.3 If $\lim_{n\to\infty} b_n = \ell_2 \neq 0$, then $\lim_{n\to\infty}\frac{1}{b_n}=\frac{1}{\lim_{n\to\infty} b_n}=\frac{1}{\ell_2}$.

Proof: By hypothesis, for any ε > 0, we can find N_1 such that $|b_n - \ell_2| < \frac{1}{2}\ell_2^2\varepsilon$ for all $n > N_1$.

Also, by the above lemma, we can find N_2 such that $|b_n| > \frac{1}{2}|\ell_2|$ for all $n > N_2$.

Thus, for any given ε > 0, we can find $N = \max\{N_1, N_2\}$ such that

$$\left|\frac{1}{b_n}-\frac{1}{\ell_2}\right|=\frac{|\ell_2-b_n|}{|b_n||\ell_2|}<\frac{\frac{1}{2}\ell_2^2\varepsilon}{|\ell_2|\cdot\frac{1}{2}|\ell_2|}=\varepsilon \text{ for all } n>N$$

Therefore, by definition, $\lim_{n\to\infty}\frac{1}{b_n}=\frac{1}{\lim_{n\to\infty} b_n}=\frac{1}{\ell_2}$.

Corollary If If $\lim_{n\to\infty} a_n = \ell_1$ and $\lim_{n\to\infty} b_n = \ell_2 \neq 0$, then $\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}=\frac{\ell_1}{\ell_2}$

Proof: By theorems 4.2 and 4.3, we have $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}(a_n)\left(\frac{1}{b_n}\right)=\left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty}\frac{1}{b_n}\right)=\ell_1\cdot\frac{1}{\ell_2}=\frac{\ell_1}{\ell_2}$.

Examples and exercises 4

Example 4.1 Evaluate each of the following, using the theorems on limits:

(a) $$\lim_{n\to\infty}\frac{3n^2-5n}{5n^2+2n-6}=\lim_{n\to\infty}\frac{3-\frac{5}{n}}{5+\frac{2}{n}-\frac{6}{n^2}}=\frac{3-\lim_{n\to\infty}\frac{5}{n}}{5+\lim_{n\to\infty}\frac{2}{n}-\lim_{n\to\infty}\frac{6}{n^2}}=\frac{3-0}{5+0-0}=\frac{3}{5}$$

(b) $$\lim_{n\to\infty}\left[\frac{n(n+2)}{n+1}-\frac{n^3}{n^2+1}\right]=\lim_{n\to\infty}\left[\frac{n(n+2)(n^2+1)-n^3(n+1)}{(n+1)(n^2+1)}\right]=\lim_{n\to\infty}\left[\frac{n^3+n^2+2n}{(n+1)(n^2+1)}\right]$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{n}+\frac{2}{n^2}}{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right)}=\frac{1+\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\frac{2}{n^2}}{\left(1+\lim_{n\to\infty}\frac{1}{n}\right)\left(1+\lim_{n\to\infty}\frac{1}{n^2}\right)}=1$$

(c) $$\lim_{n\to\infty}\left(\frac{2n-3}{3n+7}\right)^4=\lim_{n\to\infty}\left(\frac{2-\frac{3}{n}}{3+\frac{7}{n}}\right)^4=\left(\frac{2-\lim_{n\to\infty}\frac{3}{n}}{3+\lim_{n\to\infty}\frac{7}{n}}\right)^4=\left(\frac{2}{3}\right)^4=\frac{16}{81}$$

(d) $$\lim_{n\to\infty}\frac{2n^5-4n^2}{3n^7+n^3-10}=\lim_{n\to\infty}\frac{\frac{2}{n^2}-\frac{4}{n^5}}{3+\frac{1}{n^4}-\frac{10}{n^7}}=\frac{\lim_{n\to\infty}\frac{2}{n^2}-\lim_{n\to\infty}\frac{4}{n^5}}{3+\lim_{n\to\infty}\frac{1}{n^4}-\lim_{n\to\infty}\frac{10}{n^7}}=\frac{0}{3}=0$$

(e) $$\lim_{n\to\infty}\frac{1+2\cdot 10^n}{5+3\cdot 10^n}=\lim_{n\to\infty}\frac{\frac{1}{10^n}+2}{\frac{5}{10^n}+3}=\frac{\lim_{n\to\infty}\frac{1}{10^n}+2}{\lim_{n\to\infty}\frac{5}{10^n}+3}=\frac{2}{3}$$

4. Theorems on limits

Example 4.2 Find $\lim_{n\to\infty}\frac{3n^2+4n}{2n-1}$, using theorems on limits.

$$\lim_{n\to\infty}\frac{3n^2+4n}{2n-1}=\lim_{n\to\infty}n\cdot\frac{3+\frac{4}{n}}{2-\frac{1}{n}}$$

$\because$ $\lim_{n\to\infty}n=\infty$ and $\lim_{n\to\infty}\frac{3+\frac{4}{n}}{2-\frac{1}{n}}=\frac{3}{2}$

$\therefore$ $\lim_{n\to\infty}\frac{3n^2+4n}{2n-1}=\infty$

Example 4.3 Find $\lim_{n\to\infty}\frac{a_0n^m+a_1n^{m-1}+\cdots+a_m}{b_0n^m+b_1n^{m-1}+\cdots+b_m}$, where a_i and b_j are constants and m is a positive integer, Also $b_0\neq 0$.

$$\lim_{n\to\infty}\frac{a_0n^m+a_1n^{m-1}+\cdots+a_m}{b_0n^m+b_1n^{m-1}+\cdots+b_m}=\lim_{n\to\infty}\frac{a_0+\frac{a_1}{n}+\cdots+\frac{a_m}{n^m}}{b_0+\frac{b_1}{n}+\cdots+\frac{b_m}{n^m}}=\frac{a_0}{b_0}$$

Example 4.4 If $a>b>c>0$, prove that $\lim_{n\to\infty}\frac{b^n-c^n}{a^n-c^n}=0$.

$$\lim_{n\to\infty}\frac{b^n-c^n}{a^n-c^n}=\lim_{n\to\infty}\frac{\left(\frac{b}{a}\right)^n-\left(\frac{c}{a}\right)^n}{\left(\frac{a}{a}\right)^n-\left(\frac{c}{a}\right)^n}=\frac{0-0}{1-0}=0.$$

Example 4.5 Find the limit: $\lim_{n\to\infty}n\left(\sqrt{n^2+1}-n\right)$.

$$\lim_{n\to\infty}n\left(\sqrt{n^2+1}-n\right)=\lim_{n\to\infty}n\left(\sqrt{n^2+1}-n\right)\cdot\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}=\lim_{n\to\infty}n\cdot\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}=\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}+n}$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}+1}=\frac{1}{2}$$

Example 4.6 Eavluate each of the following limits.

(a) $\lim_{n\to\infty}\frac{1}{n^m}$ (where m is a positive integer.)

(b) $\lim_{n\to\infty}\frac{3n^3+n^2-n}{5n^3-1}$.

(c) $\lim_{n\to\infty}\frac{k^n+(k+1)^n}{k^{n+1}+(k+1)^{n+1}}$ (where k is a positive number.)

(d) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$.

(e) $\lim_{n\to\infty}\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)$.

Ans. (a) 0; (b) $\frac{3}{5}$; (c) $\frac{1}{k+1}$; (d) 0; (e) 0.

4. Theorems on limits

5. Squeezing principle (p.42)

5.1 Find $\lim_{n\to\infty}\frac{n}{2^n}$.

$2^n = (1+1)^n = 1 + n + \frac{n(n-1)}{2} + \cdots\cdots > \frac{n(n-1)}{2}$ for $n \ge 2$

$0 < \frac{1}{2^n} < \frac{2}{n(n-1)}$ for $n \ge 2$

$0 < \frac{n}{2^n} < \frac{2}{(n-1)}$ for $n \ge 2$

$\lim_{n\to\infty} 0 \le \lim_{n\to\infty}\frac{n}{2^n} \le \lim_{n\to\infty}\frac{2}{(n-1)} = 0$

By squeezing principle, $\lim_{n\to\infty}\frac{n}{2^n} = 0$

4.2 Find $\lim_{n\to\infty}\frac{n^{100}}{1.01^n}$.

$1.01^n = (1+0.01)^n = 1 + 0.01n + \cdots + \frac{n(n-1)\cdots(n-100)}{101!}\cdot 0.01^{101} + \cdots$

$> \frac{n(n-1)\cdots(n-100)}{101!}\cdot 0.01^{101}$ for $n \ge 101$.

$0 < \frac{1}{1.01^n} < \frac{101!}{n(n-1)\cdots(n-100)\cdot 0.01^{101}}$ for $n \ge 101$.

$0 < \frac{n^{100}}{1.01^n} < 101!\cdot\frac{n^{100}}{n(n-1)\cdots(n-100)\cdot 0.01^{101}}$ for $n \ge 101$.

$\lim_{n\to\infty} 0 \le \lim_{n\to\infty}\frac{n^{100}}{1.01^n} \le 101!\cdot\lim_{n\to\infty}\frac{1}{n\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{100}{n}\right)\cdot 0.01^{101}} = 0$

By squeezing principle, $\lim_{n\to\infty}\frac{n^{100}}{1.01^n} = 0$

4.3 Find $\lim_{n\to\infty}\frac{n!}{n^n}$.

$0 < \frac{n!}{n^n} = \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n} < \frac{1}{n}$

$\lim_{n\to\infty} 0 \le \lim_{n\to\infty}\frac{n!}{n^n} \le \lim_{n\to\infty}\frac{1}{n} = 0$

By squeezing principle, $\lim_{n\to\infty}\frac{n!}{n^n} = 0$

4.4 Prove that $\lim_{n\to\infty}\frac{a^n}{n!}=0$ for any real number a .

(a) For $a = 0$, $\lim_{n\to\infty}\frac{0^n}{n!}=0$ is obviously true.

(b) For $a > 0$, there exists $k \in \mathbf{N}$ such that $a \le k$, it follows that

$$1 \ge \frac{a}{k} > \frac{a}{k+1} > \frac{a}{k+2} > \cdots$$

When $n > k$, we have

$$0 < \frac{a^n}{n!} = \left(\frac{a}{1}\cdot\frac{a}{2}\cdot\cdots\cdot\frac{a}{k}\right)\left(\frac{a}{k+1}\cdot\frac{a}{k+2}\cdots\frac{a}{n}\right)$$

$$< \frac{a^k}{k!}\left(\frac{a}{k+1}\right)^{n-k} = \frac{a^k}{k!}\left(\frac{k+1}{a}\right)^k\left(\frac{a}{k+1}\right)^n$$

$$= \frac{(k+1)^k}{k!}\cdot\left(\frac{a}{k+1}\right)^n$$

Because k is a constant, so as $\frac{(k+1)^k}{k!}$. As $0 < \frac{a}{k+1} < 1$, so that,

$$\lim_{n\to\infty}\frac{(k+1)^k}{k!}\cdot\left(\frac{a}{k+1}\right)^n = \frac{(k+1)^k}{k!}\cdot\lim_{n\to\infty}\left(\frac{a}{k+1}\right)^n = 0$$

and hence, by squeezing principle, $\lim_{n\to\infty}\frac{a^n}{n!}=0$.

(c) For $a < 0$, let $b = -a$ (where $b > 0$), by the above result, $\lim_{n\to\infty}\frac{b^n}{n!}=0$.

$$-\frac{b^n}{n!} \le \frac{a^n}{n!} \le \frac{b^n}{n!}$$

$$0 = -\lim_{n\to\infty}\frac{b^n}{n!} \le \lim_{n\to\infty}\frac{a^n}{n!} \le \lim_{n\to\infty}\frac{b^n}{n!} = 0$$

By squeezing principle, $\lim_{n\to\infty}\frac{a^n}{n!}=0$.

We conclude that for all real number a, $\lim_{n\to\infty}\frac{a^n}{n!}=0$.

4. Theorems on limits

4.5 (a) Evaluate the limits:

(i) $\lim_{n\to\infty} 2^{\frac{1}{n}}$;

(ii) $\lim_{n\to\infty} n^{\frac{1}{n}}$.

(b) Let $0 \le r \le n$, prove that $C_{r+1}^{n+1} t^{n(r+1)} \ge C_r^n t^{(n+1)r}$ for $t \ge 1$, where C_r^n are binomial coefficients.

Hence, or otherwise, prove by induction that $\left(t^n+1\right)^{\frac{1}{n}} > \left(t^{n+1}+1\right)^{\frac{1}{n+1}}$ for $t \ge 1$.

(c) Let $x, y > 0$, $a_n = \left(x^n + y^n\right)^{\frac{1}{n}}$, prove that

(i) $\{a_n\}$ is strictly decreasing;

(ii) $\lim_{n\to\infty} a_n = \max\{x, y\}$, by using squeezing principle.

(d) Suppose $x_n \ge 0$ is a monotonic increasing sequence tends to a.

Prove that $\lim_{n\to\infty}\left(x_1^n + x_2^n + \cdots + x_n^n\right)^{\frac{1}{n}} = a$.

(a) (i) Clearly $2^{\frac{1}{n}} > 1$, otherwise $2^{\frac{1}{n}} \le 1 \Rightarrow 2 = \left(2^{\frac{1}{n}}\right)^n \le 1^n = 1 \Rightarrow 2 \le 1$!!!

Let $2^{\frac{1}{n}} = 1 + h_n$, where $h_n > 0$

$2 = (2^{\frac{1}{n}})^n = (1 + h_n)^n = 1 + nh_n + \cdots > nh_n$

$0 < h_n < \frac{2}{n}$

$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} h_n \le \lim_{n\to\infty} \frac{2}{n} = 0$

By squeezing principle, $\lim_{n\to\infty} h_n = 0 \Rightarrow \lim_{n\to\infty} 2^{\frac{1}{n}} = \lim_{n\to\infty}(1 + h_n) = 1$

(ii) Clearly $n^{\frac{1}{n}} > 1$, otherwise $n^{\frac{1}{n}} \le 1 \Rightarrow n = \left(n^{\frac{1}{n}}\right)^n \le 1^n = 1 \Rightarrow n \le 1$!!!

Let $\lim_{n\to\infty} n^{\frac{1}{n}} = 1 + k_n$, where $k_n > 0$

$n = (n^{\frac{1}{n}})^n = (1 + k_n)^n = 1 + nk_n + \frac{n(n-1)}{2}k_n^2 + \ldots > \frac{n(n-1)}{2}k_n^2$ for $n \ge 2$

$0 < k_n < \sqrt{\frac{2}{n-1}}$ for $n \ge 2$

$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} k_n \le \lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0$

By squeezing principle, $\lim_{n\to\infty} k_n = 0 \Rightarrow \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty}(1 + k_n) = 1$

(b) $C_{r+1}^{n+1} t^{n(r+1)} = \frac{(n+1)n(n-1)\cdots(n-r+1)t^{n(r+1)}}{(r+1)!}$; $C_r^n t^{(n+1)r} = \frac{n(n-1)\cdots(n-r+1)t^{(n+1)r}}{r!}$

$\frac{C_{r+1}^{n+1} t^{n(r+1)}}{C_r^n t^{(n+1)r}} = \frac{(n+1)t^{n(r+1)}}{(r+1)t^{(n+1)r}} = \frac{(n+1)}{(r+1)} t^{n-r} \ge 1$ for $t \ge 1$ and $0 \le r \le n$.

$\therefore$ $C_{r+1}^{n+1} t^{n(r+1)} \ge C_r^n t^{(n+1)r}$ for $t \ge 1$ and $0 \le r \le n$.

$(t + 1)^2 = t^2 + 2t + 1 > t^2 + 1 \Rightarrow t + 1 > \left(t^2+1\right)^{\frac{1}{2}}$. It is true for $n = 1$.

$(t^n + 1)^{n+1} = \sum_{r=0}^{n+1} C_r^{n+1} t^{nr} = 1 + \sum_{r=1}^{n+1} C_r^{n+1} t^{nr} = 1 + \sum_{r=0}^{n} C_{r+1}^{n+1} t^{n(r+1)} > \sum_{r=0}^{n} C_r^n t^{(n+1)r} = (t^{n+1} + 1)^n$

$\therefore$ $\left(t^n+1\right)^{\frac{1}{n}} > \left(t^{n+1}+1\right)^{\frac{1}{n+1}}$ for $t \ge 1$

4. Theorems on limits

(c) (i) Let $t = \frac{x}{y}$; $a_n = \left(x^n + y^n\right)^{\frac{1}{n}} = y\left[\left(\frac{x}{y}\right)^n + 1\right]^{\frac{1}{n}} = y\left(t^n + 1\right)^{\frac{1}{n}}$

$\frac{a_{n+1}}{a_n} = \frac{y\left(t^{n+1}+1\right)^{\frac{1}{n+1}}}{y\left(t^n+1\right)^{\frac{1}{n}}} = \frac{\left(t^{n+1}+1\right)^{\frac{1}{n+1}}}{\left(t^n+1\right)^{\frac{1}{n}}} < 1$ by (b)

$a_{n+1} < a_n \Rightarrow \{a_n\}$ is strictly decreasing

(ii) W.L.O.G. let $x \le y$, $y = \left(0 + y^n\right)^{\frac{1}{n}} \le \left(x^n + y^n\right)^{\frac{1}{n}} \le \left(y^n + y^n\right)^{\frac{1}{n}} = 2^{\frac{1}{n}} y$

$\lim_{n\to\infty}\left(2^{\frac{1}{n}} y\right) = y$ by $(a)(i)$

By squeezing principle, $\lim_{n\to\infty}\left(x^n + y^n\right)^{\frac{1}{n}} = y$

Similarly if $y \le x$, $\lim_{n\to\infty}\left(x^n + y^n\right)^{\frac{1}{n}} = x$

$\therefore \lim_{n\to\infty} a_n = \max\{x, y\}$

(d) $x_n = \left(x_n^n\right)^{\frac{1}{n}} \le \left(x_1^n + x_2^n + \cdots + x_n^n\right)^{\frac{1}{n}} \le \left(x_n^n + x_n^n + \cdots + x_n^n\right)^{\frac{1}{n}} = n^{\frac{1}{n}} x_n$

$a = \lim_{n\to\infty} x_n \le \lim_{n\to\infty}\left(x_1^n + x_2^n + \cdots + x_n^n\right)^{\frac{1}{n}} \le \lim_{n\to\infty} n^{\frac{1}{n}} x_n = 1(a) = a$ by (a)(ii)

5. Monotonic convergent theorem

Example 5.1 (Example of monotonic convergent theorem)

Find the limit of the sequence

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \cdots.$$

Can you prove that the limit exists?

Solution:

Let $x_1=\sqrt{2}$, $x_2=\sqrt{2+\sqrt{2}}$, $\cdots$, $x_n=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\ldots\sqrt{2}}}}}$ (there are n $\sqrt{\ }$'s and n 2's inside.)

Then $x_2=\sqrt{2+x_1}$, $x_3=\sqrt{2+x_2}$, $\cdots$, $x_n=\sqrt{2+x_{n-1}}$.

First, we try to prove that $\{x_n\}$ is a monotonic increasing sequence.

$$x_2-x_1=\sqrt{2+\sqrt{2}}-\sqrt{2}=\left(\sqrt{2+\sqrt{2}}-\sqrt{2}\right)\cdot\frac{\sqrt{2+\sqrt{2}}+\sqrt{2}}{\sqrt{2+\sqrt{2}}+\sqrt{2}}$$

$$=\frac{\left(\sqrt{2+\sqrt{2}}\right)^2-\left(\sqrt{2}\right)^2}{\sqrt{2+\sqrt{2}}+\sqrt{2}}=\frac{2+\sqrt{2}-2}{\sqrt{2+\sqrt{2}}+\sqrt{2}}=\frac{\sqrt{2}}{\sqrt{2+\sqrt{2}}+\sqrt{2}}>0$$

$\therefore x_2>x_1$

Suppose $x_k\ge x_{k-1}$.

$$x_{k+1}-x_k=\sqrt{2+x_k}-\sqrt{2+x_{k-1}}=\left(\sqrt{2+x_k}-\sqrt{2+x_{k-1}}\right)\cdot\frac{\sqrt{2+x_k}+\sqrt{2+x_{k-1}}}{\sqrt{2+x_k}+\sqrt{2+x_{k-1}}}$$

$$=\frac{2+x_k-2-x_{k-1}}{\sqrt{2+x_k}+\sqrt{2+x_{k-1}}}=\frac{x_k-x_{k-1}}{\sqrt{2+x_k}+\sqrt{2+x_{k-1}}}>0 \text{ by induction assumption}$$

$x_{k+1}>x_k$ for $k\ge 1$

By the principal of mathematical induction, $\{x_n\}$ is a monotonic increasing sequence.

$\therefore$ $\{x_n\}$ is a monotonic increasing sequence.

Claim: $x_n<2$ for all integer n.

Proof: By mathematical induction.

$n=1$, $x_1=\sqrt{2}<2$

It is true for $n=1$.

Suppose $x_k<2$ for $k>1$

Then $x_{k+1}-2=\left(\sqrt{2+x_k}-2\right)\cdot\frac{\sqrt{2+x_k}+2}{\sqrt{2+x_k}+2}$

$$=\frac{x_k-2}{\sqrt{2+x_k}+2}<0$$

$\therefore x_{k+1}<2$

So the statement is also true for $n=k+1$

By the principle of mathematical induction, $x_n<2$ for all integer n.

Since $\{x_n\}$ is an monotonic increasing sequence and is bounded above by 2, by the monotonic convergent theorem, $\lim_{n\to\infty}x_n$ exists.

5. Monotonic convergent theorem

Let $\lim_{n\to\infty} x_n = x$.

Then $x^2 = 2 + \sqrt{2 + \sqrt{2 + \sqrt{\cdots}}}$

$x^2 = 2 + x$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

As $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\cdots}}}} > 0$

So $x = -1$ is rejected

Therefore, $x = 2$ only.

6. An important limit: the number *e*

6 An important limit: the number *e*

Theorem 6.1 The sequence $\{a_n\}$ where $a_n = \left(1+\frac{1}{n}\right)^n$ is monotonic increasing and bounded above by 3, and hence is convergent.

Proof: By the binomial theorem, for any positive integer n,

$$\left(1+\frac{1}{n}\right)^n = \sum_{r=0}^{n} C_r^n \left(\frac{1}{n}\right)^r = 1+\sum_{r=1}^{n} C_r^n \left(\frac{1}{n}\right)^r$$

$$= 1+\sum_{r=1}^{n} \frac{n(n-1)\cdots(n-r+1)}{r!}\cdot\left(\frac{1}{n}\right)^r$$

$$= 1+\sum_{r=1}^{n} \frac{1}{r!}\cdot\left(1-\frac{0}{n}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)$$

$$= 1+\sum_{r=1}^{n}\left[\frac{1}{r!}\cdot\prod_{k=0}^{r-1}\left(1-\frac{k}{n}\right)\right]$$

$$< 1+\sum_{r=1}^{n}\left[\frac{1}{r!}\cdot\prod_{k=0}^{r-1}\left(1-\frac{k}{n+1}\right)\right]$$

$$< 1+\sum_{r=1}^{n+1}\left[\frac{1}{r!}\cdot\prod_{k=0}^{r-1}\left(1-\frac{k}{n+1}\right)\right]$$

$$= \left(1+\frac{1}{n+1}\right)^{n+1}$$

$\therefore$ The sequence is monotonic increasing.

Next, $\left(1+\frac{1}{n}\right)^n = 1+\sum_{r=1}^{n}\left[\frac{1}{r!}\cdot\prod_{k=0}^{r-1}\left(1-\frac{k}{n}\right)\right]$

$$< 1+\sum_{r=1}^{n}\frac{1}{r!} = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

$$< 1+1+\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)\cdot n}$$

$$= 1+1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$= 1+1+1-\frac{1}{n}$$

$$< 3$$

This shows that $\{a_n\}$ is bounded above by 3. By monotonic convergent theorem, $\{a_n\}$ converges.

The limit of this sequence is denoted by e, i.e. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Let $n \to \infty$, then $\left(1+\frac{1}{n}\right)^n = 1+\sum_{r=1}^{n}\left[\frac{1}{r!}\cdot\prod_{k=0}^{r-1}\left(1-\frac{k}{n}\right)\right] \to 1+\sum_{r=1}^{\infty}\frac{1}{r!}$

this suggests $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{k!}+\cdots = 2.71828\ldots$

6. An important limit: the number e

Corollary 6.2: For any rational number p,

(a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{pn}=e^p$

(b) $\lim_{n\to\infty}\left(1+\frac{p}{n}\right)^{n}=e^p$

Proof:

(a) $p = 0$, the result is obvious.

$p \in \mathbb{N}$. Induction on p.

$p = 1$, the result is obvious.

Suppose $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{kn}=e^k$ for some $k\in \mathbb{N}$.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{(k+1)n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{kn}\left(1+\frac{1}{n}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{kn}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e^k\cdot e=e^{(k+1)}$$

By MI, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{pn}=e^p\quad\forall\, p\in\mathbb{N}$.

If $p < 0$ and $-p \in \mathbb{N}$, let $q = -p > 0$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{pn}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-qn}=\frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{qn}}=\frac{1}{e^q}=e^p$$

If $p=\frac{1}{q}$, where $q\in\mathbb{N}$.

Claim: If $1 < x < y$, then $1<x^{\frac{1}{q}}<y^{\frac{1}{q}}$.

Proof: By contradiction, if $y^{\frac{1}{q}}\ge x^{\frac{1}{q}}$

$$\left(y^{\frac{1}{q}}\right)^q\ge\left(x^{\frac{1}{q}}\right)^q$$

$y \ge x$, which is a contradiction.

$\therefore\ 1<x^{\frac{1}{q}}<y^{\frac{1}{q}}$ …… (*)

$\because\ \left(1+\frac{1}{n}\right)^{n}$ is monotonic increasing.

$\therefore\ \left(1+\frac{1}{n}\right)^{n}\le\left(1+\frac{1}{n+1}\right)^{n+1}$

$\left(1+\frac{1}{n}\right)^{\frac{n}{q}}\le\left(1+\frac{1}{n+1}\right)^{\frac{n+1}{q}}$ (by the result of (*))

$\therefore\ \left(1+\frac{1}{n}\right)^{\frac{n}{q}}$ is monotonic increasing.

Moreover, $\left(1+\frac{1}{n}\right)^{n}<3$

$\left(1+\frac{1}{n}\right)^{\frac{n}{q}}<3^{\frac{1}{q}}$ (by the result of (*))

6. An important limit: the number e

$\left(1+\frac{1}{n}\right)^{\frac{n}{q}}$ is bounded above by $3^{\frac{1}{q}}$.

By monotonic convergent theorem, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{q}}$ exists. Let $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{q}} = \ell$.

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{\frac{n}{q}}\right]^q = e = \ell^q$$

$$\therefore \ell = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{q}}$$

When $p = \frac{m}{q}$, where $q \neq 0$, $m, q \in \mathbb{N}$. and $(m, q) = 1$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{pn} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{mn}{q}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{\frac{n}{q}}\right]^m = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{q}}\right]^m = \left(e^{\frac{1}{q}}\right)^m = e^{\frac{m}{q}} = e^p$$

When $p = -\frac{m}{q}$, where $q \neq 0$, $m, q \in \mathbb{N}$. and $(m, q) = 1$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{pn} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-\frac{mn}{q}} = \frac{1}{\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{\frac{mn}{q}}\right]} = \frac{1}{e^{\frac{m}{q}}} = e^{-\frac{m}{q}} = e^p$$

The theorem is proved.

(b) To prove $\lim_{n\to\infty}\left(1+\frac{p}{n}\right)^n = e^p$

$p = 0$, the result is obvious.

$p \in \mathbb{N}$, Induction on p.

$p = 1$, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1$; it is true for $p = 1$.

Suppose $\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$, $k \in \mathbb{N}$

$$\lim_{n\to\infty}\left(1+\frac{k+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{k}{n+1}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \lim_{n+1\to\infty}\frac{\left(1+\frac{k}{n+1}\right)^{n+1}}{\left(1+\frac{k}{n+1}\right)}$$

$$= e \times \frac{e^p}{1} = e^{p+1}$$

$\therefore$ It is also true for $p = k + 1$

By the principal of mathematical induction, $\lim_{n\to\infty}\left(1+\frac{p}{n}\right)^n = e^p$ for $p \in \mathbb{N}$

6. An important limit: the number e

$p < 0$, let $q = -p > 0$, $\lim_{n\to\infty}\left(1+\frac{p}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{q}{n}\right)^n$

$$= \frac{1}{\lim_{n\to\infty}\left(\frac{n}{n-q}\right)^n} \text{, valid for } n > q$$

$$= \frac{1}{\lim_{n\to\infty}\left(1+\frac{q}{n-q}\right)^{n-q}} \cdot \frac{1}{\lim_{n\to\infty}\left(1+\frac{q}{n-q}\right)^{q}}$$

$$= \frac{1}{e^q} = e^p$$

$p = \frac{1}{q}$, where $q \in \mathbb{N}$. Let $m = nq$.

$$\lim_{n\to\infty}\left(1+\frac{p}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{nq}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{m}{q}} = e^{\frac{m}{q}} = e^p \quad \text{(by the result of (a))}$$

When $p = \frac{m}{q}$, where $q \neq 0$, $m \in \mathbb{Z}$, $q \in \mathbb{N}$. and $(m, q) = 1$

$$\lim_{n\to\infty}\left(1+\frac{p}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{m}{nq}\right)^n = \lim_{k\to\infty}\left(1+\frac{m}{k}\right)^{\frac{k}{q}} \text{, where } k = nq \in \mathbb{N}.$$

$$= \left[\lim_{k\to\infty}\left(1+\frac{m}{k}\right)^k\right]^{\frac{1}{q}} = \left(e^m\right)^{\frac{1}{q}} = e^p \quad \text{(by the above result and the result of } (a))$$

The theorem is proved.

6. An important limit: the number e

Example 6.1 Advanced Level Pure Mathematics Calculus and Analytical Geometry I by K.S. Ng, Y. K.kwok, p.90 Exercise 2D Q2(b)

(i) $\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^n = \frac{1}{\lim_{n\to\infty}\left(\frac{3n}{3n-1}\right)^n} = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{3n-1}\right)^{\frac{3n-1}{3}+\frac{1}{3}}} = \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{3n-1}\right)^{\frac{3n-1}{3}}\lim_{n\to\infty}\left(1+\frac{1}{3n-1}\right)^{\frac{1}{3}}}$

$= \frac{1}{\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{m}{3}}\cdot 1}$; $m = 3n - 1$

$= \frac{1}{e^{\frac{1}{3}}} = e^{-\frac{1}{3}}$

Remark $\left(1+\frac{1}{m}\right)^m \le \left(1+\frac{1}{m+1}\right)^{m+1}$

$\left(1+\frac{1}{m}\right)^{\frac{m}{3}} \le \left(1+\frac{1}{m+1}\right)^{\frac{m+1}{3}}$

$\therefore$ It is a monotonic increasing sequence.

Moreover, $\left(1+\frac{1}{m}\right)^m < 3$

$\therefore \left(1+\frac{1}{m}\right)^{\frac{m}{3}} < 3^{\frac{1}{3}} \Rightarrow$ the sequence is bounded above by $3^{\frac{1}{3}}$.

By monotonic convergent theorem, $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{m}{3}} = \ell$

$\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{m}{3}}\right]^3 = \ell^3 \Rightarrow e = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = \ell^3 \Rightarrow \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{m}{3}} = e^{\frac{1}{3}}$

(ii) $\lim_{n\to\infty}\left(1-\frac{4}{2n-3}\right)^{2-n} = \lim_{n\to\infty}\left(\frac{2n-7}{2n-3}\right)^{2-n} = \lim_{n\to\infty}\left(\frac{2n-3}{2n-7}\right)^{n-2} = \lim_{n\to\infty}\left(1+\frac{4}{2n-7}\right)^{\frac{2n-7}{2}+\frac{3}{2}}$

$= \lim_{n\to\infty}\left(1+\frac{4}{2n-7}\right)^{\frac{2n-7}{2}} \cdot \lim_{n\to\infty}\left(1+\frac{4}{2n-7}\right)^{\frac{3}{2}} = \lim_{m\to\infty}\left(1+\frac{4}{m}\right)^{\frac{m}{2}} \cdot 1$; $m = 2n - 7$

$= \lim_{m\to\infty}\left[\left(1+\frac{1}{m}\right)\left(1+\frac{1}{m+1}\right)\left(1+\frac{1}{m+2}\right)\left(1+\frac{1}{m+3}\right)\right]^{\frac{m}{2}}$

$= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{\frac{m}{2}} \lim_{m\to\infty}\left(1+\frac{1}{m+1}\right)^{\frac{m}{2}} \lim_{m\to\infty}\left(1+\frac{1}{m+2}\right)^{\frac{m}{2}} \lim_{m\to\infty}\left(1+\frac{1}{m+3}\right)^{\frac{m}{2}}$

$= e^{\frac{1}{2}} \cdot e^{\frac{1}{2}} \cdot e^{\frac{1}{2}} \cdot e^{\frac{1}{2}} = e^2$

(iii) $\lim_{n\to\infty}\left(1+\frac{2}{n}-\frac{8}{n^2}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{4}{n}\right)\left(1-\frac{2}{n}\right)\right]^n = \lim_{n\to\infty}\left(1+\frac{4}{n}\right)^n \lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n = e^4 \cdot e^{-2} = e^2$

(iv) $\lim_{n\to\infty}\left(1-\frac{3}{4n}-\frac{5}{8n^2}\right)^n = \lim_{n\to\infty}\left[\left(1-\frac{5}{4n}\right)\left(1+\frac{1}{2n}\right)\right]^n = \lim_{n\to\infty}\left(1-\frac{5}{4n}\right)^n \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = e^{-\frac{5}{4}} \cdot e^{\frac{1}{2}} = e^{-\frac{3}{4}}$

6. An important limit: the number e

Example 6.2

$$\lim_{n\to\infty}\left(\frac{n^2+1}{n^2}\right)^{n^2+1}=\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2+1}=\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2}\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{1}=e$$

Example 6.3

Let λ be fixed real number and k is a positive integer, find $\lim_{n\to\infty}\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$.

$$\begin{aligned}\lim_{n\to\infty}\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}&=\lim_{n\to\infty}\frac{1}{k!}\cdot\frac{n(n-1)\cdots(n-k+1)}{n^k}\cdot\lambda^k\cdot\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}\\&=\lim_{n\to\infty}\frac{1}{k!}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\cdots\cdot\frac{n-k+1}{n}\cdot\lambda^k\cdot\frac{e^{-\lambda}}{1}\\&=\lim_{n\to\infty}\frac{1}{k!}\cdot1\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\cdot\lambda^k\cdot e^{-\lambda}\\&=\frac{1}{k!}\lambda^k e^{-\lambda}\end{aligned}$$

Remark: $\lim_{n\to\infty}\left(1+\frac{\lambda}{n}\right)^n=e^\lambda\quad\forall\lambda\in\mathbb{R}$.

Proof: $\forall\lambda\in\mathbb{R}$, we can find a sequence of rational numbers $\{q_m\}$ so that $\lim_{m\to\infty}q_m=\lambda$.

$\lim_{m\to\infty}\lim_{n\to\infty}\left(1+\frac{q_m}{n}\right)^n=\lim_{m\to\infty}e^{q_m}=e^{\lim_{n\to\infty}q_m}=e^\lambda$ (We have assumed that $f(x)=e^x$ is continuous.)

Example 6.4

(a) Prove the following inequalities:

(i) $n!<\left(\frac{n+1}{2}\right)^n$ for $n>1$

(ii) $\left(\frac{n}{e}\right)^n<n!<e\left(\frac{n}{2}\right)^n$

where e is the limit of the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$.

(b) Using (a)(ii) to prove that $\lim_{n\to\infty}\frac{n!}{n^n}=0$.

(a) (i) By the inequality of the means, we have

$\sqrt{1\cdot n}<\frac{1}{2}(n+1)$

$\sqrt{2\cdot(n-1)}<\frac{1}{2}[2+(n-1)]=\frac{1}{2}(n+1)$

$\sqrt{3\cdot(n-2)}<\frac{1}{2}[3+(n-2)]=\frac{1}{2}(n+1)$

……………………………

$\sqrt{n\cdot1}<\frac{1}{2}(n+1)$

Multiplying, $n!<\left(\frac{n+1}{2}\right)^n$.

6. An important limit: the number e

(ii) By (i), $n! < \left(\frac{n+1}{2}\right)^n = \left(\frac{n}{2}\right)^n\left(1+\frac{1}{n}\right)^n < e\left(\frac{n}{2}\right)^n$.

As $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is an increasing sequence with e as its limit, so that

$\left(1+\frac{1}{n}\right)^n < e$ for n = 1, 2, ⋯

$\left(1+\frac{1}{1}\right) < e$, $\left(1+\frac{1}{2}\right)^2 < e$, $\left(1+\frac{1}{3}\right)^3 < e$, ⋯, $\left(1+\frac{1}{n}\right)^n < e$

Multiplying $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)^2\left(1+\frac{1}{3}\right)^3\cdots\left(1+\frac{1}{n}\right)^n < e^n$

$\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n < e^n$

$\frac{(n+1)^n}{n!} < e^n$

$\left(\frac{n}{e}\right)^n < \frac{(n+1)^n}{e^n} < n! < e\left(\frac{n}{2}\right)^n$

(b) By (a)(ii), $\frac{1}{e^n}\cdot n^n < n! < \frac{e}{2^n}\cdot n^n$

$\frac{1}{e^n} < \frac{n!}{n^n} < \frac{e}{2^n}$

As $n \to \infty$, $\frac{1}{e^n} \to 0$, $\frac{e}{2^n} \to 0$

By squeezing principle, $\lim_{n\to\infty}\frac{n!}{n^n} = 0$.

Example 6.5

Prove that $x_n = \left(1+\frac{1}{n}\right)^n$ is monotonic increasing and bounded above. While the sequence $y_n = \left(1+\frac{1}{n}\right)^{n+1}$ is monotonic decreasing and bounded below.

Hence show that they have the same limit: $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = e$.

[Hint: Consider $\frac{y_n}{y_{n+1}}$ and use Bernoulli inequality: $(1 + x)^n \ge 1 + nx$ for $x > -1$.]

$$\frac{x_{n+1}}{x_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \left[\frac{n(n+2)}{(n+1)^2}\right]^n\left(1+\frac{1}{n+1}\right) = \left[1-\frac{1}{(n+1)^2}\right]^n\left(1+\frac{1}{n+1}\right)$$

$$\ge \left[1-\frac{n}{(n+1)^2}\right]\left(1+\frac{1}{n+1}\right) = \left[\frac{n^2+n+1}{(n+1)^2}\right]\left(\frac{n+2}{n+1}\right) = \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1} > 1$$

$\therefore x_{n+1} > x_n$

The fact that x_n is bounded above by 3 has already been proved.

$\therefore \lim_{n\to\infty} x_n$ exists.

6. An important limit: the number e

$$\frac{y_n}{y_{n+1}}=\frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}}=\left[\frac{(n+1)^2}{n(n+2)}\right]^{n+1}\left(\frac{n+1}{n+2}\right)=\left[1+\frac{1}{n(n+2)}\right]^{n+1}\left(\frac{n+1}{n+2}\right)$$

$$\geq\left[1+\frac{n+1}{n(n+2)}\right]\left(\frac{n+1}{n+2}\right)=\left[\frac{n^2+3n+1}{n(n+2)}\right]\cdot\frac{n+1}{n+2}=\frac{n^3+4n^2+4n+1}{n^3+4n^2+4n}>1$$

$\therefore y_n > y_{n+1}$, so it is monotonic decreasing.

Clearly y_n is bounded below by 0

$\therefore \lim_{n\to\infty} y_n$ exists.

Let $\lim_{n\to\infty} x_n = p$, $\lim_{n\to\infty} y_n = q$

$$q=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=\lim_{n\to\infty}x_n\cdot 1=\lim_{n\to\infty}x_n=p$$

Example 6.6

(a) Let $a_n=\left(1+\frac{1}{n}\right)^n$, $b_n=\left(1+\frac{1}{n}\right)^{n+1}$, $n = 1, 2, \cdots$

Without reference to binomial theorem, show that $\{a_n\}$ is monotonic increasing and $\{b_n\}$ is monotonic decreasing.

Hence, determine which is larger number $(1000000)^{1000000}$ or $(1000001)^{999999}$.

(b) From the results of (a) show that $\left(\frac{n}{e}\right)^n < n! < e(n+1)\left(\frac{n}{e}\right)^n$.

For $n > 6$, derive the sharper inequality $n! < n\left(\frac{n}{e}\right)^n$.

(a) Let $y=\left(1+\frac{1}{x}\right)^x$.

$$\ln y = x\ln\left(1+\frac{1}{x}\right)$$

$$y'=\left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right)+\frac{x}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\right]$$

$$=\left(1+\frac{1}{x}\right)^x\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]$$

Let $z=\ln\left(1+\frac{1}{x}\right)-\frac{1}{1+x}$

$$z'=\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{(1+x)^2}$$

$$=-\frac{1}{x(1+x)}+\frac{1}{(1+x)^2}=\frac{-1-x+x}{x(1+x)^2}=-\frac{1}{x(1+x)^2}<0$$

$\therefore z$ is strictly decreasing

$$\therefore \forall x>0,\ z(x)>\lim_{x\to\infty}z(x)=\lim_{x\to\infty}\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]=\ln 1-0=0$$

$\therefore z > 0$

$y' > 0$

6. An important limit: the number e

y is increasing

$a_n = \left(1+\frac{1}{n}\right)^n$ is monotonic increasing.

Let $y = \left(1+\frac{1}{x}\right)^{x+1}$.

$\ln y = (x + 1)\ln\left(1+\frac{1}{x}\right)$

$y' = \left(1+\frac{1}{x}\right)^{x+1}\left[\ln\left(1+\frac{1}{x}\right)+\frac{x+1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\right]$

$= \left(1+\frac{1}{x}\right)^{x+1}\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{x}\right]$

Let $z = \ln\left(1+\frac{1}{x}\right)-\frac{1}{x}$

$z' = \frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{x^2}$

$= -\frac{1}{x(1+x)}+\frac{1}{x^2} \qquad = \frac{-x+1+x}{x^2(1+x)} = \frac{1}{x(1+x)^2} > 0$

$\therefore$ z is strictly increasing

$\therefore$ $\forall x > 0$, $z(x) < \lim_{x\to\infty} z(x) = \lim_{x\to\infty}\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{x}\right] = \ln 1 - 0 = 0$

$\therefore z < 0$

$y' < 0$

y is decreasing

$b_n = \left(1+\frac{1}{n}\right)^{n+1}$ is monotonic decreasing.

$(n + 1) > e > \left(1+\frac{1}{n}\right)^n$ it is true for $n > 1$

$n\left(1+\frac{1}{n}\right) > \left(1+\frac{1}{n}\right)^n$

$n > \left(1+\frac{1}{n}\right)^{n-1}$

$n > \left(\frac{n+1}{n}\right)^{n-1}$

$n^n > (n+1)^{n-1}$

Put $n = 1000000$, then $(1000000)^{1000000} > (1000001)^{999999}$.

(b) The fact that $\left(\frac{n}{e}\right)^n < n!$ has been proved in **Example 14** (a)(ii).

$\because$ $b_n \searrow$ to e

$\therefore$ $e < \left(1+\frac{1}{1}\right)^2$, $e < \left(1+\frac{1}{2}\right)^3$, $e < \left(1+\frac{1}{3}\right)^4$, … , $e < \left(1+\frac{1}{n}\right)^{n+1}$

6. An important limit: the number e

Multiplying $e^n < \left(1+\frac{1}{1}\right)^2\left(1+\frac{1}{2}\right)^3\left(1+\frac{1}{3}\right)^4\cdots\left(1+\frac{1}{n}\right)^{n+1}$

$e^n < \left(\frac{2}{1}\right)^2\left(\frac{3}{2}\right)^3\left(\frac{4}{3}\right)^4\cdots\left(\frac{n+1}{n}\right)^{n+1}$

$e^n < \frac{(n+1)^{n+1}}{n!}$

$\therefore \frac{1}{e^n} > \frac{n!}{(n+1)^{n+1}}$

$e(n+1)\left(\frac{n}{e}\right)^n > e(n+1)\frac{n!n^n}{(n+1)^{n+1}} = e\left(\frac{n}{n+1}\right)^n n! > \left(1+\frac{1}{n}\right)^n\left(\frac{n}{n+1}\right)^n n! = n!$

$\therefore \left(\frac{n}{e}\right)^n < n! < e(n+1)\left(\frac{n}{e}\right)^n$

To prove that for $n > 6$, $n! < n\left(\frac{n}{e}\right)^n$.

Induction on n.

When $n = 7$, L.H.S. = 7! = 5040, R.H.S. $= 7\left(\frac{7}{e}\right)^7 = 5257$

$\therefore$ L.H.S. < R.H.S., it is true for $n = 7$

Suppose $k! < k\left(\frac{k}{e}\right)^k$ for some positive integer $k > 6$.

$\because b_n \searrow$ to e

$\therefore e < \left(1+\frac{1}{k}\right)^{k+1}$

$e\,k^{k+1} < (k+1)^{k+1}$

$(k+1)! = (k+1)k! < (k+1)\cdot\frac{k^{k+1}}{e^k} < \frac{(k+1)^{k+2}}{e^{k+1}}$; by MI, the statement is true for $n > 6$.