Theory on summation of numbers

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1. $r(r+1)-(r-1)r=2r$

   $1\times2-0\times1=2\times1$

   $2\times3-1\times2=2\times2$

   …………………

   $n\times(n+1)-(n-1)\times n=2\times n$

   Add these equations together, $n(n+1)=2\times(1+2+\ldots+n)$

   $\Rightarrow\ 1+2+\cdots+n=\frac{1}{2}n(n+1)$

2. $r(r+1)(r+2)-(r-1)r(r+1)=3r(r+1)$

   $1\times2\times3-0\times1\times2=3\times1\times2$

   $2\times3\times4-1\times2\times3=3\times2\times3$

   ………………………………………

   $n(n+1)(n+2)-(n-1)n(n+1)=3n(n+1)$

   Add these equations together, $n(n+1)(n+2)=3\times[1\times2+2\times3+\cdots+n\times(n+1)]$

   $\Rightarrow\ 1\times2+2\times3+\cdots+n\times(n+1)=\frac{1}{3}n(n+1)(n+2)$

3. Using a similar technique, we have

   $1\times2\times3+2\times3\times4+\cdots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$

   If m is a positive integer, we can use mathematical induction to prove that

   $1\times2\times\cdots\times m+2\times3\times\cdots\times(m+1)+\cdots+n(n+1)\cdots(n+m-1)=\frac{1}{m+1}n(n+1)\cdots(n+m)$

4. $\frac{1}{r+1}-\frac{1}{r}=-\frac{1}{r(r+1)}$

   $\frac{1}{2}-\frac{1}{1}=-\frac{1}{1\times2}$

   $\frac{1}{3}-\frac{1}{2}=-\frac{1}{2\times3}$

   ……………….

   $\frac{1}{n+1}-\frac{1}{n}=-\frac{1}{n(n+1)}$

   Add these equations together, $\frac{1}{n+1}-\frac{1}{1}=-\left[\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{n(n+1)}\right]$

   $\frac{1}{1\times2}+\frac{1}{2\times3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}=\frac{n}{n+1}$

5. $\frac{1}{(r+1)(r+2)}-\frac{1}{r(r+1)}=-\frac{2}{r(r+1)(r+2)}$

$\frac{1}{2\times 3}-\frac{1}{1\times 2}=-\frac{2}{1\times 2\times 3}$

$\frac{1}{3\times 4}-\frac{1}{2\times 3}=-\frac{2}{2\times 3\times 4}$

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$\frac{1}{(n+1)(n+2)}-\frac{1}{n(n+1)}=-\frac{2}{n(n+1)(n+2)}$

Add these equations together,

$\frac{1}{(n+1)(n+2)}-\frac{1}{2}=-2\left[\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}\right]$

$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]=\frac{n^2+3n}{4(n+1)(n+2)}$

6. $\frac{1}{(r+1)(r+2)(r+3)}-\frac{1}{r(r+1)(r+2)}=-\frac{3}{r(r+1)(r+2)(r+3)}$

Using a similar technique,

$\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\cdots+\frac{1}{n(n+1)(n+2)(n+3)}=\frac{1}{3}\left[\frac{1}{6}-\frac{1}{(n+1)(n+2)(n+3)}\right]$

If m is a positive integer, we can use mathematical induction to prove that

$\frac{1}{1\cdot 2\cdots m}+\frac{1}{2\cdot 3\cdots(m+1)}+\cdots+\frac{1}{n(n+1)\cdots(n+m-1)}=\frac{1}{m-1}\left[\frac{1}{(m-1)!}-\frac{1}{(n+1)(n+2)\cdots(n+m-1)}\right]$

7. $r^2 = r(r+1) - r$

$\sum_{r=1}^{n} r^2=\sum_{r=1}^{n} r(r+1)-\sum_{r=1}^{n} r$

$\sum_{r=1}^{n} r^2=\frac{1}{3}n(n+1)(n+2)-\frac{1}{2}n(n+1)$

$= n(n+1)\left[\frac{1}{3}(n+2)-\frac{1}{2}\right]$

$= n(n+1)\left(\frac{n}{3}+\frac{1}{6}\right)$

$= \frac{1}{6}n(n+1)(2n+1)$

Method 2

$(r+1)^3 - r^3 = 3r^2 + 3r + 1$

$$\sum_{r=1}^{n}\left[(r+1)^3 - r^3\right] = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + n$$

$$(n+1)^3 - 1 = 3\sum_{r=1}^{n} r^2 + \frac{3n(n+1)}{2} + n$$

$$3\sum_{r=1}^{n} r^2 = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n$$

$$3\sum_{r=1}^{n} r^2 = \frac{n}{2}\left(2n^2 + 6n + 6 - 3n - 3 - 2\right)$$

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}\left(2n^2 + 3n + 1\right)$$

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

8. $r^3 = r(r+1)(r+2) - 3r(r+1) + r$

$$\sum_{r=1}^{n} r^3 = \sum_{r=1}^{n} r(r+1)(r+2) - 3\sum_{r=1}^{n} r(r+1) + \sum_{r=1}^{n} r$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n(n+1)(n+2)(n+3) - n(n+1)(n+2) + \frac{1}{2}n(n+1)$$

$$= n(n+1)\left[\frac{1}{4}(n+2)(n+3) - (n+2) + \frac{1}{2}\right]$$

$$= \frac{1}{4}n(n+1)\left(n^2 + 5n + 6 - 4n - 8 + 2\right)$$

$$= \frac{1}{4}n^2(n+1)^2$$

Method 2

$(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$

$$\sum_{r=1}^{n}\left[(r+1)^4 - r^4\right] = 4\sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r + n$$

$$(n+1)^4 - 1 = 4\sum_{r=1}^{n} r^3 + n(n+1)(2n+1) + 2n(n+1) + n$$

$$n^4 + 4n^3 + 6n^2 + 4n = 4\sum_{r=1}^{n} r^3 + 2n^3 + 3n^2 + n + 2n^2 + 2n + n$$

$$4\sum_{r=1}^{n} r^3 = n^4 + 2n^3 + n^2$$

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

9. $r^4 = r(r+1)(r+2)(r+3) - 6r(r+1)(r+2) + 7r(r+1) - r$

$$\sum_{r=1}^{n} r^4 = \sum_{r=1}^{n} r(r+1)(r+2)(r+3) - 6\sum_{r=1}^{n} r(r+1)(r+2) + 7\sum_{r=1}^{n} r(r+1) - \sum_{r=1}^{n} r$$

$$\sum_{r=1}^{n} r^4 = \frac{1}{5}n(n+1)(n+2)(n+3)(n+4) - \frac{3}{2}n(n+1)(n+2)(n+3) + \frac{7}{3}n(n+1)(n+2) - \frac{1}{2}n(n+1)$$

$$= \frac{1}{30}n(n+1)[6(n+2)(n+3)(n+4) - 45(n+2)(n+3) + 70(n+2) - 15]$$

$$= \frac{1}{30}n(n+1)(6n^3 + 9n^2 + n - 1) = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1)$$

Method 2

$$(r+1)^5 - r^5 = 5r^4 + 10r^3 + 10r^2 + 5r + 1$$

$$\sum_{r=1}^{n}\left[(r+1)^5 - r^5\right] = 5\sum_{r=1}^{n} r^4 + 10\sum_{r=1}^{n} r^3 + 10\sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r + n$$

$$(n+1)^5 - 1 = 5\sum_{r=1}^{n} r^4 + \frac{5n^2(n+1)^2}{2} + \frac{5n(n+1)(2n+1)}{3} + \frac{5n(n+1)}{2} + n$$

$$n^5 + 5n^4 + 10n^3 + 10n^2 + 5n = 5\sum_{r=1}^{n} r^4 + \frac{n(n+1)(15n^2 + 15n + 20n + 10 + 15)}{6} + n$$

$$n^5 + 5n^4 + 10n^3 + 10n^2 + 4n = 5\sum_{r=1}^{n} r^4 + \frac{n(n+1)(15n^2 + 35n + 25)}{6}$$

$$n(n^4 + 5n^3 + 10n^2 + 10n + 4) = 5\sum_{r=1}^{n} r^4 + \frac{n(n+1)(15n^2 + 35n + 25)}{6}$$

$$n(n+1)(n^3 + 4n^2 + 6n + 4) = 5\sum_{r=1}^{n} r^4 + \frac{n(n+1)(15n^2 + 35n + 25)}{6}$$

$$5\sum_{r=1}^{n} r^4 = \frac{n(n+1)}{6}(6n^3 + 24n^2 + 36n + 24 - 15n^2 - 35n - 25)$$

$$\sum_{r=1}^{n} r^4 = \frac{n(n+1)}{30}(6n^3 + 9n^2 + n - 1)$$

$$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1)$$

Method 3

Given $1^n + 2^n + \cdots + x^n = S_n(x)$

$x^n = S_n(x) - S_n(x-1)$
$nx^{n-1} = S_n'(x) - S_n'(x-1)$

Example Find the value of $\frac{1}{1\times2\times4}+\frac{1}{2\times3\times5}+\cdots+\frac{1}{97\times98\times100}$.

Let $\frac{1}{r\times(r+1)\times(r+3)}\equiv\frac{A}{r}+\frac{B}{(r+1)}+\frac{C}{(r+3)}$.

$1 \equiv A(r+1)(r+3)+Br(r+3)+Cr(r+1)$

Put $r = 0$, $1 = 3A \Rightarrow A=\frac{1}{3}$

Put $r = -1$, $1 = -2B \Rightarrow B=-\frac{1}{2}$

Put $r = -3$, $1 = 6C \Rightarrow C=\frac{1}{6}$

$$\frac{1}{r\times(r+1)\times(r+3)}\equiv\frac{1}{3r}-\frac{1}{2(r+1)}+\frac{1}{6(r+3)}\equiv\frac{1}{3}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\frac{1}{6}\left(\frac{1}{r+1}-\frac{1}{r+3}\right)$$

$$\frac{1}{1\times2\times4}+\frac{1}{2\times3\times5}+\cdots+\frac{1}{97\times98\times100}$$

$$=\sum_{r=1}^{r=97}\left[\frac{1}{3}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\frac{1}{6}\left(\frac{1}{r+1}-\frac{1}{r+3}\right)\right]$$

$$=\frac{1}{3}\sum_{r=1}^{r=97}\left(\frac{1}{r}-\frac{1}{r+1}\right)-\frac{1}{6}\sum_{r=1}^{r=97}\left(\frac{1}{r+1}-\frac{1}{r+3}\right)$$

$$=\frac{1}{3}\left(1-\frac{1}{98}\right)-\frac{1}{6}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{99}-\frac{1}{100}\right)$$

$$=\frac{7}{36}+\frac{199}{59400}-\frac{1}{294}$$

$$=\frac{565801}{2910600}$$