Apollonius's Theorem a theorem about parallelogram

Created by Mr. Francis Hung Last updated: February 13, 2024

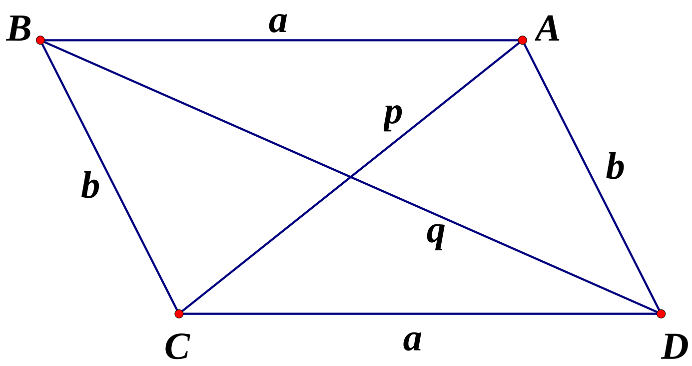
Let $ABCD$ is a parallelogram.

Let $AB = CD = a$; $AD = BC = b$.

Let $AC = p$; $BD = q$.

Then $2(a^2 + b^2) = p^2 + q^2$

Proof: Let H, and K be points on DC (produced) such that $BK \perp CD$, $AH \perp CD$



Then $\Delta AHD \cong \Delta BKC$ (AAS)

Let $BK = AH = h$; $CK = HD = t$.

In ΔAHD, $h^2 + t^2 = b^2$ $\cdots$ (1)

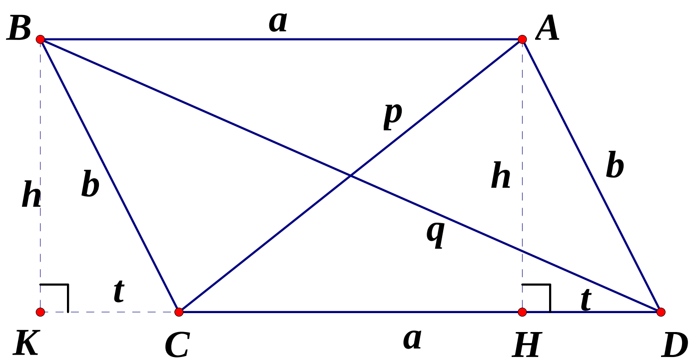
In ΔAHC, $h^2 + (a - t)^2 = p^2$ $\cdots$ (2)

In ΔBKD, $h^2 + (a + t)^2 = q^2$ $\cdots$ (3)

(2) + (3): $2h^2 + 2a^2 + 2t^2 = p^2 + q^2$

Sub. (1): $2a^2 + 2b^2 = p^2 + q^2$

Hence the theorem is proved.



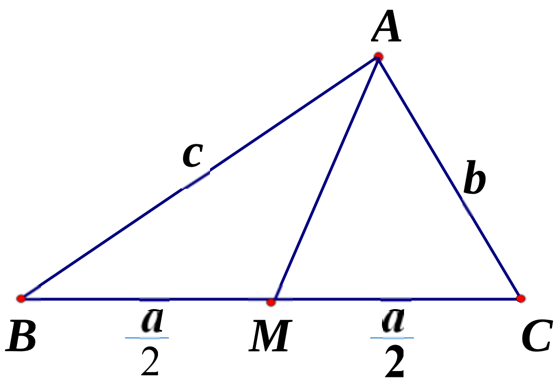
The **median** of a triangle ABC.

In ΔABC, let $BC = a$, $AC = b$, $AB = c$.

Let M be the mid point of BC.

Then the line AM is called a median.

$$AM = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$



Proof:

Produce AM to D so that $AM = MD$.

$ABDC$ is a //-gram. (diags bisect each other)

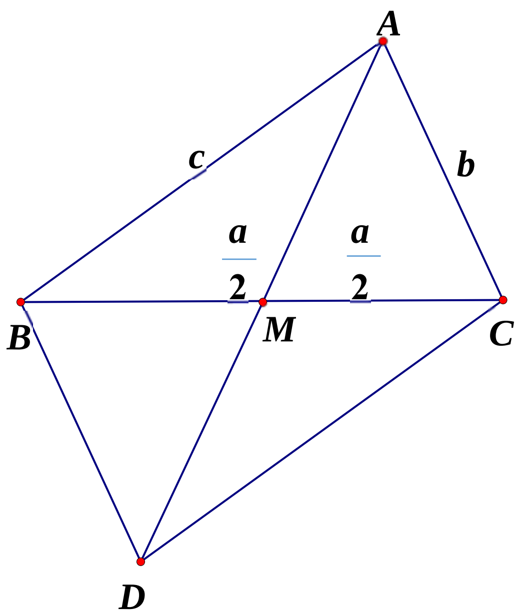
$BD = AC = b$, $CD = AB = c$

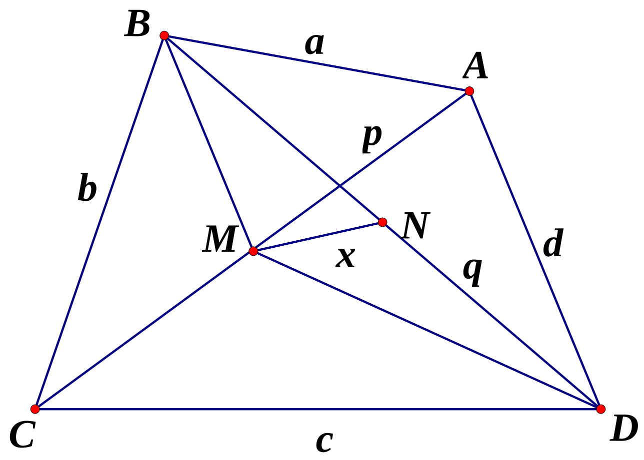
(opp. sides of //-gram)

By the above Apollonius Theorem,

$2(b^2 + c^2) = a^2 + (2\ AM)^2$

$$AM = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \cdots (4)$$





Let *ABCD* be a quadrilateral.

Let $AB = a$, $BC = b$, $CD = b$, $DA = d$;

Let $AC = p$, $BD = q$.

Let *M* and *N* be the mid points of *AC* and *BD* respectively. $MN = x$.

In general, $a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4x^2$

If $a^2 + b^2 + c^2 + d^2 = p^2 + q^2$,

then *ABCD* is a parallelogram.

Proof:

By the above theorem on median,

In ΔABC, $BM^2 = \dfrac{2a^2 + 2b^2 - p^2}{4}$ $\cdots$ (5)

In ΔADC, $DM^2 = \dfrac{2c^2 + 2d^2 - p^2}{4}$ $\cdots$ (6)

In ΔBMD, $MN^2 = \dfrac{2BM^2 + 2DM^2 - q^2}{4}$

Sub (5) and (6):

$$MN^2 = x^2 = \frac{a^2 + b^2 - \frac{p^2}{2} + c^2 + d^2 - \frac{p^2}{2} - q^2}{4} = \frac{a^2 + b^2 + c^2 + d^2 - p^2 - q^2}{4}$$

$\therefore$ $a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4x^2$ $\cdots$ (7) (The result due to Casey 1888)

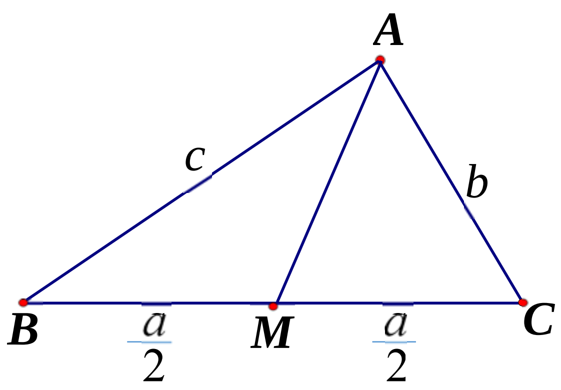
If $a^2 + b^2 + c^2 + d^2 = p^2 + q^2$, then $MN = x = 0$

Therefore, $M = N$ and hence the two diagonals bisect each other at *M* (= *N*).

ABCD is a parallelogram

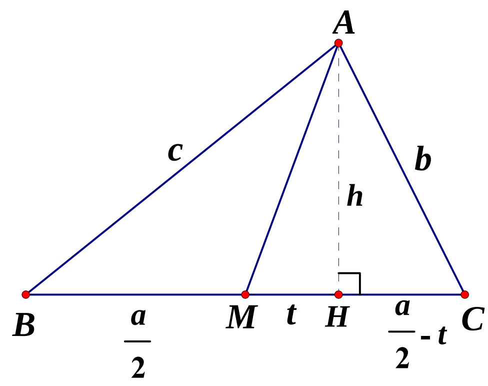
The **median** of a triangle ABC.
In ΔABC, let $BC = a$, $AC = b$, $AB = c$.
Let M be the mid point of BC.
Then the line AM is called a median.

$AM = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$



Proof: Draw $AH \perp BC$. Let $AH = h$.

Let $MH = t$, then $CH = \frac{a}{2} - t$, $BH = \frac{a}{2} + t$



In ΔAMH, $h^2 + t^2 = AM^2$ $\cdots$ (1)

In ΔAHC, $h^2 + \left(\frac{a}{2} - t\right)^2 = b^2$ $\cdots$ (2)

In ΔABH, $h^2 + \left(\frac{a}{2} + t\right)^2 = c^2$ $\cdots$ (3)

(2) + (3): $2h^2 + \frac{a^2}{2} + 2t^2 = b^2 + c^2$

Sub. (1): $2AM^2 + \frac{a^2}{2} = b^2 + c^2$

$4AM^2 = 2b^2 + 2c^2 - a^2$

$AM = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$