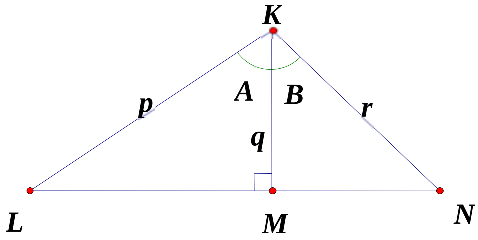
Compound angled formulae

Created by Mr. Francis Hung on 20090205 Last updated: March 19, 2023

The formula sin(*A* + *B*) = sin *A* cos *B* + cos *A* sin *B*

0° < A < 90°, 0° < B < 90°

Consider a triangle *KLN* with a right angle at *M*, as shown in the figure.



Let $\angle LKM = A$, $\angle MKN = B$, then $\angle LKN = A + B$.

Area of $\Delta LKM = \frac{1}{2} pq \sin A$

Area of $\Delta MKN = \frac{1}{2} qr \sin B$

Area of $\Delta LKN = \frac{1}{2} pr \sin(A+B)$

Area of ΔLKM + Area of ΔMKN = Area of ΔLKN

$\therefore\ \frac{1}{2} pq \sin A + \frac{1}{2} qr \sin B = \frac{1}{2} pr \sin(A+B)$

$\sin(A+B) = \frac{q}{r}\sin A + \frac{q}{p}\sin B$

sin(*A* + *B*) = sin *A* cos *B* + cos *A* sin *B*

Method 2 0° < A < 90°, 0° < B < 90°, 0° < A + B < 90°

In ΔPQN, $\angle PNQ = 90°$, $\angle NPQ = A$

PQRS is a rectangle. $PR = r$, $PQ = q$, $PN = p$.

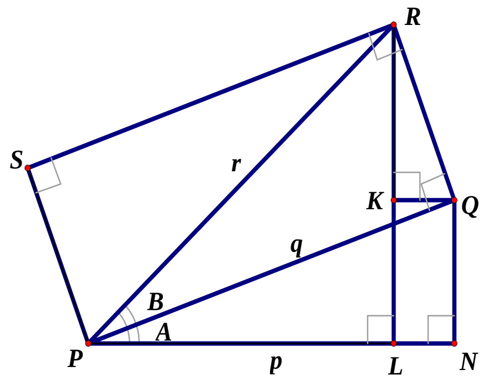
$\angle RPQ = B$, $\angle RPN = A + B$.

$RL \perp PN$, $KQ \perp RL$.

$\angle KQP = A$ (alt. ∠s KQ // PN)

$\angle RQK = 90° - A$

$\angle KRQ = A$ (∠ sum of Δ)



KQNL is a rectangle

$RL = RK + KL = RQ \cos A + QN$

$r \sin (A + B) = r \sin B \cos A + q \sin A$

$r \sin (A + B) = r \sin B \cos A + r \cos B \sin A$

$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B$

Method 3 In ΔABC, $A + B + C = 180°$

By sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \Rightarrow \sin A = \frac{a}{2R}$; $\sin B = \frac{b}{2R}$; $\sin C = \frac{c}{2R}$

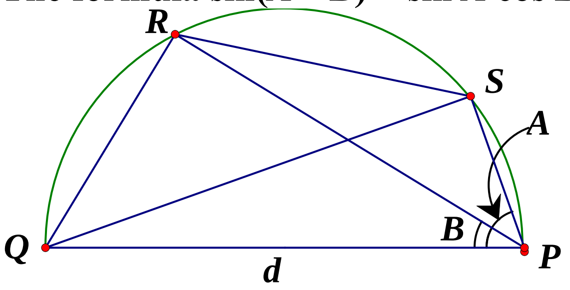
By cosine rule, $\cos A = \frac{b^2+c^2-a^2}{2bc}$; $\cos B = \frac{a^2+c^2-b^2}{2ac}$

$$\sin A \cos B + \cos A \sin B = \frac{a}{2R}\cdot\frac{a^2+c^2-b^2}{2ac} + \frac{b^2+c^2-a^2}{2bc}\cdot\frac{b}{2R}$$

$$= \frac{2c^2}{4cR} = \frac{c}{2R} = \sin C = \sin(180° - (A + B)) \text{ (∠s sum of Δ)}$$

$$= \sin (A + B)$$

The formula sin(A – B) = sin A cos B – cos A sin B



Suppose $PQRS$ is a semi-circle, with diameter $PQ = d$, $\angle QPS = A$, $\angle QPR = B$.
$\angle PRQ = 90°$, $\angle PSQ = 90°$ ($\angle$ in semi-circle)
$QR = d \sin B$
$QS = d \sin A$
$PR = d \cos B$
$PS = d \cos A$
$\angle RPS = A - B$

By Sine rule on ΔPRS, $\dfrac{RS}{\sin(A-B)} = \dfrac{PS}{\sin\angle PRS}$

$\dfrac{RS}{\sin(A-B)} = \dfrac{PS}{\sin\angle PQS}$ ($\because \angle PRS = \angle PQS$, $\angle$s in the same segment)

$\dfrac{RS}{\sin(A-B)} = \dfrac{d\cos A}{\sin(90^\circ - A)}$ ($\because \angle PQS = 90° - \alpha$, $\angle$s sum of ΔPQS)

$RS = d \sin(A - B)$ ($\because \sin(90° - A) = \cos A$)

By Ptolemy's theorem, i.e. $PR \cdot QS = RS \cdot PQ + QR \cdot PS$
$d \cos B \cdot d \sin A = d \sin(A - B) \cdot d + d \sin B \cdot d \cos A$
$\Rightarrow \sin(A - B) = \sin A \cos B - \cos A \sin B$

Method 2 New Trend Additional Mathematics Volume One (2002) p. 142
Consider a triangle KLN with a right angle at N, as shown in the figure.
Let $\angle LKN = A$, $\angle MKN = B$, then $\angle LKM = A - B$.

Area of $\Delta LKM = \dfrac{1}{2} qr \sin(A-B)$

Area of $\Delta MKN = \dfrac{1}{2} pq \sin B$

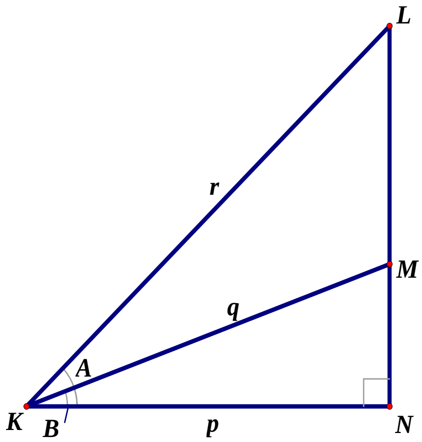
Area of $\Delta LKN = \dfrac{1}{2} pr \sin A$

Area of ΔLKM + Area of ΔMKN = Area of ΔLKN

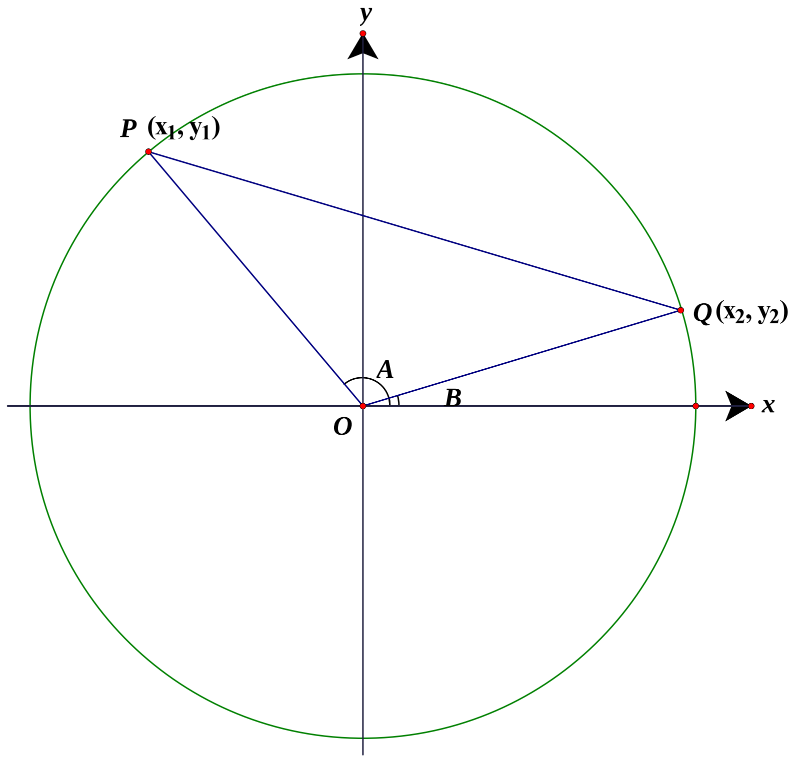
$\therefore \dfrac{1}{2} qr \sin(A-B) + \dfrac{1}{2} pq \sin B = \dfrac{1}{2} pr \sin A$

$\sin(A-B) = \dfrac{p}{q}\sin A - \dfrac{p}{r}\sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$



The formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$



Draw a unit circle with centre O and radius 1. Suppose $P(x_1, y_1)$, $Q(x_2, y_2)$ are two points on the circumference. Suppose OP makes an angle A with positive x-axis. OQ makes an angle B with positive axis. $\angle QOP = A - B$.

$x_1 = \cos A$, $y_1 = \sin A$; $x_2 = \cos B$, $y_2 = \sin B$

By cosine rule, $PQ^2 = OP^2 + OQ^2 - 2OP\cdot OQ \cos \angle POQ$.

$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 1 + 1 - 2 \cos (A - B)$

$(\cos A - \cos B)^2 + (\sin A - \sin B)^2 = 1 + 1 - 2 \cos (A - B)$

$\cos^2 A - 2 \cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B = 2 - 2 \cos (A - B)$

$-2(\cos A \cos B + \sin A \sin B) = -2 \cos (A - B)$

$\therefore \cos (A - B) = \cos A \cos B + \sin A \sin B \cdots\cdots (1)$

Variation We have already proved the formula $\cos(A-B) = \cos A\cos B + \sin A\sin B$, where A and B can be positive or negative real numbers.

Replace B by $-B$, then $\cos(A+B) = \cos A\cos(-B) + \sin A\sin(-B)$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$ ⋯⋯ (2)

Replace A by $90° - A$, then $\sin(A-B) = \cos[90° - (A-B)] = \cos[(90° - A) + B]$

$= \cos(90° - A)\cos B - \sin(90° - A)\sin B$

Hence, $\sin(A-B) = \sin A\cos B - \cos A\sin B$ ⋯⋯ (3)

Replace B by $-B$ again, then $\sin(A+B) = \sin A\cos(-B) - \cos A\sin(-B)$

So, $\sin(A+B) = \sin A\cos B + \cos A\sin B$ ⋯⋯ (4)

Now, $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$

$$= \frac{\dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B - \sin A\sin B}{\cos A\cos B}}$$

$\therefore\ \tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ ⋯⋯ (5)

Replace B by $-B$ again, then $\tan(A-B) = \dfrac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)} = \dfrac{\tan A - \tan B}{1 + \tan A + \tan B}$ ⋯⋯ (6)

Example 1 Find the value of sin 15°, cos 15° and tan 15°.

$\sin 15° = \sin(60° - 45°) = \sin 60°\cos 45° - \cos 60°\sin 45° = \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} - \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

$\cos 15° = \cos(60° - 45°) = \cos 60°\cos 45° + \sin 60°\sin 45° = \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

$\tan 15° = \tan(60° - 45°) = \dfrac{\tan 60° - \tan 45°}{1 + \tan 60°\tan 45°} = \dfrac{\sqrt{3}-1}{1+\sqrt{3}} = \dfrac{(\sqrt{3}-1)}{(1+\sqrt{3})}\cdot\dfrac{(\sqrt{3}-1)}{(\sqrt{3}-1)} = \dfrac{3-2\sqrt{3}+1}{3-1} = 2-\sqrt{3}$

Example 2 Find the value of sin 75°, cos 75° and tan 75°.

$\sin 75° = \cos 15° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ ($\because \cos(90° - \theta) = \sin\theta$)

$\cos 75° = \sin 15° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

$\tan 75° = \dfrac{1}{\tan 15°} = \dfrac{1}{2-\sqrt{3}} = \dfrac{1}{2-\sqrt{3}}\cdot\dfrac{2+\sqrt{3}}{2+\sqrt{3}} = 2+\sqrt{3}$

Example 3 Find the value of sin 165°, cos 165° and tan 165°.

$\sin 165° = \sin(180° - 15°) = -\sin 15° = -\dfrac{\sqrt{6}-\sqrt{2}}{4}$

$\cos 165° = \cos(180° - 15°) = -\cos 15° = -\dfrac{\sqrt{6}+\sqrt{2}}{4}$

$\tan 165° = \tan(180° - 15°) = -\tan 15° = -\left(2-\sqrt{3}\right)$

Using these results, we can deduce the values of the sines, cosines and tangents of 15°, 75°, 105°, 165°, 195°, 255°, 285° and 345°, or equivalently in radians $\dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{7\pi}{12}, \dfrac{11\pi}{12}, \dfrac{13\pi}{12}, \dfrac{17\pi}{12}, \dfrac{19\pi}{12}\ \dfrac{23\pi}{12}$.

Example 4 If $A + B = 45°$, prove that $(1 + \tan A)(1 + \tan B) = 2$.
Hence find the value of $(1 + \tan 1°)(1 + \tan 2°)\cdots(1 + \tan 44°)(1 + \tan 45°)$.

$\tan(A + B) = \tan 45° = 1$

$\dfrac{\tan A+\tan B}{1-\tan A\tan B}=1$

$\tan A + \tan B = 1 - \tan A \tan B$

$1 + \tan A + \tan B + \tan A \tan B = 2$

$(1 + \tan A)(1 + \tan B) = 2$

$(1 + \tan 1°)(1 + \tan 44°) = 2$

$(1 + \tan 2°)(1 + \tan 43°) = 2$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (there are 22 pairs of $(1 + \tan A)(1 + \tan B)$)

$1 + \tan 45° = 2$

$(1 + \tan 1°)(1 + \tan 2°)\cdots(1 + \tan 44°)(1 + \tan 45°) = 2^{23}$.

$$\cot(A+B)=\frac{1}{\tan\left(A+B\right)}=\frac{1-\tan A\tan B}{\tan A+\tan B}=\frac{\cot A\cot B-1}{\cot A+\cot B}\ \cdots\cdots\ (7)$$

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}\ \cdots\cdots\ (8)$$

Example 5 (Identity) If $A + B + C = 180°$, prove that $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

$$\text{L.H.S.} = \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$$

$$=\cot\frac{A}{2}+\cot\frac{B}{2}+\tan(90°-\frac{C}{2})=\cot\frac{A}{2}+\cot\frac{B}{2}+\tan\frac{180^\circ-C}{2}$$

$$=\cot\frac{A}{2}+\cot\frac{B}{2}+\tan\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\frac{A}{2}+\cot\frac{B}{2}+\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}$$

$$=\frac{\cot\frac{A}{2}+\cot\frac{B}{2}-\tan\frac{B}{2}-\tan\frac{A}{2}+\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\cot\frac{A}{2}\cot\frac{B}{2}\cdot\frac{\cot\frac{A}{2}+\cot\frac{B}{2}}{\cot\frac{A}{2}\cot\frac{B}{2}-1}$$

$$=\cot\frac{A}{2}\cot\frac{B}{2}\cdot\frac{1}{\cot\frac{A+B}{2}}=\cot\frac{A}{2}\cot\frac{B}{2}\cdot\tan\frac{A+B}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cdot\tan\left(90^\circ-\frac{C}{2}\right)$$

$$=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\text{R.H.S.}$$

Example 6 If $\sin A + \cos B = p$, $\cos A - \sin B = q$ and $A - B = 30°$, prove that $p^2 + q^2 - 3 = 0$

$\sin A = p - \cos B$

$\cos A = q + \sin B$

$\therefore\ 1 = (p - \cos B)^2 + (q + \sin B)^2$

$p^2 + q^2 + 2q \sin B - 2p \cos B = 0$

$p^2 + q^2 + 2[(\cos A - \sin B) \sin B - (\sin A + \cos B) \cos B] = 0$

$p^2 + q^2 - 2 + 2(\sin B \cos A - \cos B \sin A) = 0$

$p^2 + q^2 - 2 + 2 \sin(B - A) = 0$

$p^2 + q^2 - 3 = 0$ ($\because A - B = 30°$)

Example 7 Evaluate tan θ tan (θ – 60°) + tan θ tan (θ + 60°) + tan (θ – 60°) tan (θ + 60°)

Expression = tan θ [tan (θ – 60°) + tan (θ + 60°)] + tan (θ – 60°) tan (θ + 60°)

$$=\tan\theta\cdot\left(\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}+\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}\right)+\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}\times\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}$$

$$=\tan\theta\cdot\frac{\tan\theta-\sqrt{3}-\sqrt{3}\tan^2\theta+3\tan\theta+\tan\theta+\sqrt{3}+\sqrt{3}\tan^2\theta+3\tan\theta}{1-3\tan^2\theta}+\frac{\tan^2\theta-3}{1-3\tan^2\theta}$$

$$=\tan\theta\cdot\frac{8\tan\theta}{1-3\tan^2\theta}+\frac{\tan^2\theta-3}{1-3\tan^2\theta}=\frac{9\tan^2\theta-3}{1-3\tan^2\theta}$$

$$=\frac{3\left(3\tan^2\theta-1\right)}{1-3\tan^2\theta}=-3$$

Example 8 Prove that tan nθ – tan(n – 1)θ = tan θ [1 + tan nθ tan (n – 1)θ]

Use this formula to find $\sum_{r=1}^{n}\tan r\theta\tan\left(r-1\right)\theta$.

tan θ = tan[nθ – (n – 1)θ] $=\frac{\tan n\theta-\tan\left(n-1\right)\theta}{1+\tan n\theta\tan\left(n-1\right)\theta}$

∴ tan nθ – tan(n – 1)θ = tan θ [1 + tan nθ tan (n – 1)θ]

$$\sum_{r=1}^{n}\tan r\theta\tan\left(r-1\right)\theta=\frac{1}{\tan\theta}\sum_{r=1}^{n}\tan\theta\left[1+\tan r\theta\tan\left(r-1\right)\theta\right]-n$$

$$=\frac{1}{\tan\theta}\sum_{r=1}^{n}\left[\tan r\theta-\tan\left(r-1\right)\theta\right]-n=\frac{\tan n\theta}{\tan\theta}-n$$

Example 9 Solve 12 sin θ – 5 cos θ = 13 for 0° ≤ θ ≤ 360°.

(12 sin θ – 5 cos θ)^2 = 169

144 sin^2 θ – 120 sin θ cos θ + 25 cos^2 θ = 169(sin^2 θ + cos^2 θ)

25 sin^2 θ + 120 sin θ cos θ + 144 cos^2 θ = 0

25 tan^2 θ + 120 tan θ + 144 = 0

(5 tan θ + 12)^2 = 0

tan θ = $-\frac{12}{5}$

θ = 112.6°, 292.6°

Check: when θ = 112.6°,

LHS = 12 sin 112.6° – 5 cos 112.6° = 13 = RHS

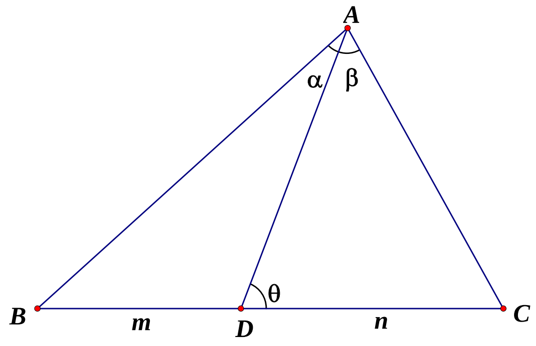
When θ = 292.6°,

LHS = 12 sin 292.6° – 5 cos 292.6° = –13 ≠ RHS

∴ θ = 112.6° only

Classwork 1 Solve the equation 3 sin θ – 2 cos θ = 1 for 0° ≤ θ ≤ 360°. [θ = 49.79° or 197.59°]

Example 10 The point D divides the sides BC of ΔABC internally so that $BD : DC = m : n$. If $\angle BAD = \alpha$, $\angle CAD = \beta$ and $\angle CDA = \theta$, prove that $m \cot \alpha - n \cot \beta = (m + n) \cot \theta = n \cot B - m \cot C$.



In ΔADC, $\dfrac{n}{\sin\beta} = \dfrac{AC}{\sin\theta} \quad \cdots (1)$

In ΔABC, $\dfrac{m+n}{\sin(\alpha+\beta)} = \dfrac{AC}{\sin B}$

$\angle B = \theta - \alpha$ (ext. $\angle$ of ΔABD)

$$\frac{m+n}{\sin(\alpha+\beta)} = \frac{AC}{\sin(\theta-\alpha)} \quad \cdots (2)$$

$(1)\div(2)$ $\dfrac{n}{\sin\beta} \times \dfrac{\sin(\alpha+\beta)}{(m+n)} = \dfrac{AC}{\sin\theta} \times \dfrac{\sin(\theta-\alpha)}{AC}$

$m \sin \theta \sin(\alpha + \beta) = (m + n) \sin (\theta - \alpha) \sin \beta$

$m \sin \theta \sin(\alpha + \beta) = (m + n) \sin \beta (\sin \theta \cos \alpha - \cos \theta \sin \alpha)$

$(m + n) \sin \beta \cos \theta \sin \alpha = (m + n) \sin \beta \sin \theta \cos \alpha - m \sin \theta \sin(\alpha + \beta)$

$(m + n) \sin \alpha \sin \beta \cos \theta = [(m + n) \sin \beta \cos \alpha - m \sin(\alpha + \beta)] \sin \theta$

$(m + n) \sin \alpha \sin \beta \cos \theta = [(m + n) \cos \alpha \sin \beta - m \sin \alpha \cos \beta - m \cos \alpha \sin \beta] \sin \theta$

$(m + n) \sin \alpha \sin \beta \cos \theta = (n \cos \alpha \sin \beta - m \sin \alpha \cos \beta) \sin \theta$

$\therefore (m + n) \cot \theta = \dfrac{m\cos\alpha\sin\beta - n\sin\alpha\cos\beta}{\sin\alpha\sin\beta}$

$= m \cot \alpha - n \cot \beta$

In ΔABD, $\dfrac{m}{\sin\alpha} = \dfrac{AB}{\sin(180^\circ-\theta)}$

$\alpha = \theta - B$ (ext. $\angle$ of ΔABD)

$\therefore$ $\dfrac{m}{\sin(\theta-B)} = \dfrac{AB}{\sin\theta} \quad \cdots (3)$

In ΔABC, $\dfrac{m+n}{\sin A} = \dfrac{AB}{\sin C}$

$$\frac{m+n}{\sin(180^\circ-(B+C))} = \frac{AB}{\sin C}$$

$$\frac{m+n}{\sin(B+C)} = \frac{AB}{\sin C} \quad \cdots (4)$$

$(3)\div(4)$ $\dfrac{m}{\sin(\theta-B)} \times \dfrac{\sin(B+C)}{(m+n)} = \dfrac{AB}{\sin\theta} \times \dfrac{\sin C}{AB}$

$m \sin \theta \sin(B + C) = (m + n) \sin C \sin (\theta - B)$

$m \sin \theta \sin(B + C) = (m + n) \sin C (\sin \theta \cos B - \cos \theta \sin B)$

$(m + n) \sin C \cos \theta \sin B = (m + n) \sin C \sin \theta \cos B - m \sin \theta \sin(B + C)$

$(m + n) \sin B \sin C \cos \theta = [(m + n) \cos B \sin C - m \sin(B + C)] \sin \theta$

$(m + n) \sin B \sin C \cos \theta = [(m + n) \cos B \sin C - m \sin B \cos C - m \cos B \sin C] \sin \theta$

$(m + n) \sin B \sin C \cos \theta = (n \cos B \sin C - m \sin B \cos C) \sin \theta$

$\therefore (m + n) \cot \theta = \dfrac{n\cos B\sin C - m\sin B\cos C}{\sin B\sin C}$

$\therefore (m + n) \cot \theta = n \cot B - m \cot C$

Subsidiary angles

Example 11

(a) Rewrite the expression $4\cos\theta - 3\sin\theta$ in the form $r\cos(\theta+\alpha)$, where $r > 0$ and $0 \le \alpha < \frac{\pi}{2}$.

Give your answer correct to 2 decimal places.

(b) Find the maximum and minimum values of $4\cos\theta - 3\sin\theta$.

(a) $r\cos(\theta+\alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

$= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$

As $4\cos\theta - 3\sin\theta = (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$

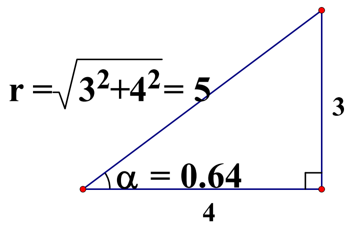
$r\cos\alpha = 4$ and $r\sin\alpha = 3$

$r = \sqrt{3^2+4^2} = 5$

$\tan\alpha = \frac{3}{4}$

$\alpha = 0.64$ radians (correct to 2 decimal places)

$\therefore 4\cos\theta - 3\sin\theta = 5\cos(\theta + 0.64)$



(b) Let $y = 4\cos\theta - 3\sin\theta$

$= 5\cos(\theta+\alpha)$,

Since $-1 \le \cos(\theta+\alpha) \le 1$,

$\therefore -5 \le 5\cos(\theta+\alpha) \le 5$

$\therefore$ The maximum value of y is 5.

The minimum value of y is -5.

Classwork 2

(*a*) Express $3\sin\theta - 2\cos\theta$ in the form $R\sin(\theta-\alpha)$ where $R > 0$ and $0° \le \alpha \le 90°$.

(*b*) Hence, or otherwise, find the range of possible values of $3\sin\theta - 2\cos\theta$.

[$\sqrt{13}\sin\left(\theta - 33.69°\right), -\sqrt{13} \le 3\sin\theta - 2\cos\theta \le \sqrt{13}$]

Example 12 Solve the equation $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$, where $0° \le \theta \le 360°$

Consider a right-angled triangle as shown in the figure.

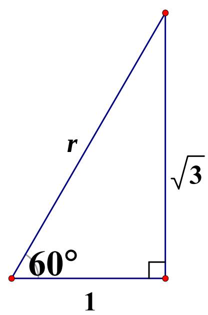
$r^2 = 3 + 1$

$r = 2$

and $\tan\alpha = \sqrt{3}$

$\alpha = 60°$

$2\sin 60° = \sqrt{3}$, $2\cos 60° = 1$



The equation becomes

$2\sin 60°\cos\theta + 2\cos 60°\sin\theta = \sqrt{2}$

$2\sin(60° + \theta) = \sqrt{2}$

$\sin(60° + \theta) = \frac{\sqrt{2}}{2}$

$60° + \theta = 135°$ or $405°$

$\theta = 75°$ or $345°$

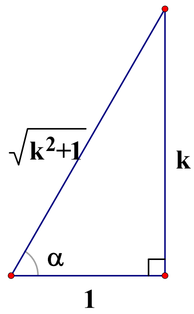
Classwork 1 Solve the equation $3\sin\theta - 2\cos\theta = 1$ for $0° \le \theta \le 360°$. [$\theta = 49.79°$ or $197.59°$]

Example 13 Show that if the equation $\sin\theta + k\cos\theta = \sqrt{2}+1$ has a solution then $k^2 \geq 2\sqrt{2}+2$

$$\sqrt{k^2+1}\cdot\left(\sin\theta\cdot\frac{1}{\sqrt{k^2+1}}+\cos\theta\cdot\frac{k}{\sqrt{k^2+1}}\right)=\sqrt{2}+1$$

$$\sin\theta\cdot\cos\alpha+\cos\theta\cdot\sin\alpha=\frac{\sqrt{2}+1}{\sqrt{k^2+1}}$$

$$\sin(\theta+\alpha)=\frac{\sqrt{2}+1}{\sqrt{k^2+1}}$$



The trigonometric equation has a solution, so $-1\leq\frac{\sqrt{2}+1}{\sqrt{k^2+1}}\leq 1$

$$\left(\sqrt{2}+1\right)^2\leq\left(\sqrt{k^2+1}\right)^2$$

$$2+2\sqrt{2}+1\leq k^2+1$$

$$k^2\geq 2\sqrt{2}+2$$

Example 14 Prove that, for all real x, the maximum value of $c\sin(x+A)+d\sin(x+B)$ is $\sqrt{c^2+d^2+2cd\cos(A-B)}$.

$c\sin(x+A)+d\sin(x+B)$

$= c\sin x\cos A + c\cos x\sin A + d\sin x\cos B + d\cos x\sin B$

$= \sin x(c\cos A + d\cos B) + \cos x(c\sin A + d\sin B)$

$$= u\cdot\left[\sin x\cdot\frac{(c\cos A+d\cos B)}{u}+\cos x\cdot\frac{(c\sin A+d\sin B)}{u}\right] \quad \cdots\cdots (*)$$

where $u=\sqrt{(c\cos A+d\cos B)^2+(c\sin A+d\sin B)^2}$

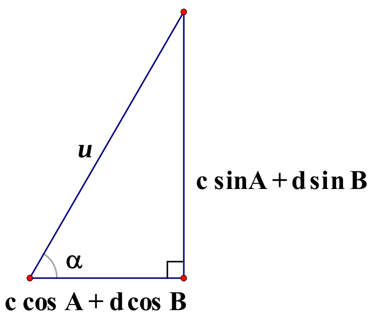
$$=\sqrt{\left(c^2\cos^2 A+2cd\cos A\cos B+d^2\cos^2 B\right)+\left(c^2\sin^2 A+2cd\sin A\sin B+d^2\sin^2 B\right)}$$

$$=\sqrt{\left(c^2\cos^2 A+c^2\sin^2 A\right)+\left(d^2\cos^2 B+d^2\sin^2 B\right)+\left(2cd\cos A\cos B+2cd\sin A\sin B\right)}$$

$$=\sqrt{c^2+d^2+2cd\cos(A-B)}$$

(*) $c\sin(x+A)+d\sin(x+B) = u\cdot(\sin x\cos\alpha+\cos x\sin\alpha)$

$= u\sin(x+\alpha)$



$\therefore$ The maximum value of $c\sin(x+A)+d\sin(x+B)$ is $u=\sqrt{c^2+d^2+2cd\cos(A-B)}$.

Double angle formulae

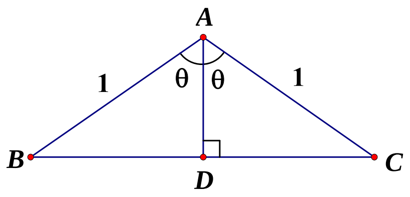
$\because$ sin $(A + B)$ = sin A cos B + cos A sin B

Let $B = A$

sin $(A + A)$ = sin A cos A + cos A sin A

sin $2A$ = 2 sin A cos A $\cdots$ (9)

Note: The above formula can be proved by the following method:
Given an isosceles triangle ABC with $AB = AC = 1$, let $\angle BAC = 2\theta$.



Draw $AD \perp BC$, where D is the foot of perpendicular from A to BC. Then $\Delta ABD \cong \Delta ACD$ (R.H.S.)

$\angle BAD = \angle CAD = \theta$. (corr. $\angle$s $\cong\Delta$'s)

$AD = 1\times\cos\theta = \cos\theta$

$BD = CD = 1\times\sin\theta = \sin\theta$

By finding the area of triangle in two different ways,

$$\frac{1}{2}AB\times AC\sin 2\theta = \frac{1}{2}BC\times AD$$

$\therefore$ sin 2θ = 2 sin θ cos θ

$\because$ cos $(A + B)$ = cos A cos B – sin A sin B

Let $B = A$

cos $(A + A)$ = cos A cos A – sin A sin A

$\therefore$ cos $2A$ = $\cos^2 A - \sin^2 A$ $\cdots$ (10)

By using the identity $\sin^2 A + \cos^2 A = 1$

So $\sin^2 A = 1 - \cos^2 A$

Sub. into (10): $\cos 2A = \cos^2 A - (1 - \cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$= 2\cos^2 A - 1$

$\therefore$ $\cos 2A = 2\cos^2 A - 1$ $\cdots$ (11)

Also, $\cos^2 A = 1 - \sin^2 A$

Sub. into (10): $\cos 2A = (1 - \sin^2 A) - \sin^2 A$

$= 1 - 2\sin^2 A$

$\therefore$ $\cos 2A = 1 - 2\sin^2 A$ $\cdots$ (12)

$$\because\ \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Let $B = A$

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A\tan A}$$

$$\therefore\ \tan 2A = \frac{2\tan A}{1 - \tan^2 A} \cdots (13)$$

$$\cot 2A = \frac{1}{\tan 2A} = \frac{1 - \tan^2 A}{2\tan A} = \frac{1}{2}\left(\cot A - \frac{1}{\cot A}\right) \cdots (14)$$

Example 15 Without using calculators, evaluate cos 36° cos 72°.

$$\cos 36^\circ \cos 72^\circ = \frac{2\sin 36^\circ \cos 36^\circ \cos 72^\circ}{2\sin 36^\circ}$$

$$= \frac{2\sin 72^\circ \cos 72^\circ}{4\sin 36^\circ}$$

$$= \frac{\sin 144^\circ}{4\sin 36^\circ} = \frac{1}{4}$$

Example 16 Without using calculators, evaluate cos 12° cos 24° cos 48° cos 96°.

$$\cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 96^\circ = \frac{2\sin 12^\circ \cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 96^\circ}{2\sin 12^\circ}$$

$$= \frac{2\sin 24^\circ \cos 24^\circ \cos 48^\circ \cos 96^\circ}{4\sin 12^\circ}$$

$$= \frac{2\sin 48^\circ \cos 48^\circ \cos 96^\circ}{8\sin 12^\circ}$$

$$= \frac{2\sin 96^\circ \cos 96^\circ}{16\sin 12^\circ}$$

$$= \frac{\sin 192^\circ}{16\sin 12^\circ} = -\frac{1}{16}$$

Classwork 3

(a) By mathematical induction, prove that for all positive integer *n*,

$$\cos\theta\cos 2\theta\cos 4\theta\cdots\cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n\sin\theta}.$$

(b) Hence evaluate cos 20° cos 40° cos 80° without using calculators.

Example 17 Prove that $\dfrac{\tan 2\theta+\sec 2\theta-1}{\tan 2\theta-\sec 2\theta+1}=\tan\left(\theta+\dfrac{\pi}{4}\right)$.

$$\text{L.H.S.}=\frac{\tan 2\theta+\sec 2\theta-1}{\tan 2\theta-\sec 2\theta+1}$$

$$=\frac{\left(\frac{\sin 2\theta}{\cos 2\theta}+\frac{1}{\cos 2\theta}-1\right)}{\left(\frac{\sin 2\theta}{\cos 2\theta}-\frac{1}{\cos 2\theta}+1\right)}\cdot\frac{\cos 2\theta}{\cos 2\theta}$$

$$=\frac{\sin 2\theta+1-\cos 2\theta}{\sin 2\theta-1+\cos 2\theta}$$

$$=\frac{2\sin\theta\cos\theta+1-\left(1-2\sin^2\theta\right)}{2\sin\theta\cos\theta-1+\left(1-2\sin^2\theta\right)}$$

$$=\frac{2\sin\theta\cos\theta+2\sin^2\theta}{2\sin\theta\cos\theta-2\sin^2\theta}$$

$$=\frac{2\sin\theta\left(\cos\theta+\sin\theta\right)}{2\sin\theta\left(\cos\theta-\sin\theta\right)}$$

$$=\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$$

$$\text{R.H.S.}=\tan\left(\theta+\frac{\pi}{4}\right)$$

$$=\frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\cdot\tan\frac{\pi}{4}}$$

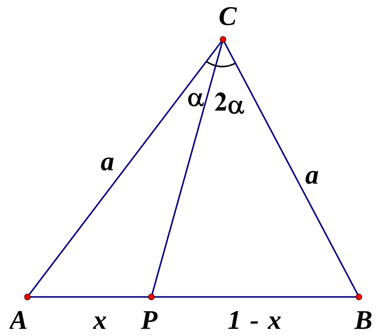
$$=\frac{\tan\theta+1}{1-\tan\theta\cdot 1}$$

$$=\frac{\left(\frac{\sin\theta}{\cos\theta}+1\right)}{\left(1-\frac{\sin\theta}{\cos\theta}\right)}\cdot\frac{\cos\theta}{\cos\theta}$$

$$=\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$$

$\therefore$ L.H.S. = R. H.S.

Example 18 In the given figure, ΔABC is isosceles with $AC = BA = a$, $AB = 1$. P is a point on AB such that $\angle ACP = \alpha$, $\angle BCP = 2\alpha$, show that $AP = \dfrac{1}{1+2\cos\alpha}$. Hence deduce that $\dfrac{1}{3} < AP < \dfrac{1}{2}$.



Let $AP = x$, then $BP = 1 - x$.

In ΔAPC, $\dfrac{x}{\sin\alpha} = \dfrac{CP}{\sin A}$ $\cdots$ (1)

In ΔBPC, $\dfrac{1-x}{\sin 2\alpha} = \dfrac{CP}{\sin B}$ $\cdots$ (2)

$\angle A = \angle B$ (base $\angle$s isos. Δ)

$\therefore$ (1) = (2): $\dfrac{x}{\sin\alpha} = \dfrac{1-x}{\sin 2\alpha}$

$\dfrac{x}{\sin\alpha} = \dfrac{1-x}{2\sin\alpha\cos\alpha}$

$2x\cos\alpha = 1 - x$

$x = AP = \dfrac{1}{1+2\cos\alpha}$

$0° < \angle C = 3\alpha < 180°$

$\therefore$ $0° < \alpha < 60°$

$\dfrac{1}{2} < \cos\alpha < 1$

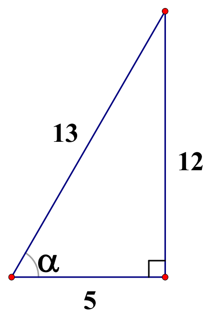
$1 < 2\cos\alpha < 2$

$2 < 1 + 2\cos\alpha < 3$

$\dfrac{1}{2} > \dfrac{1}{1+2\cos\alpha} > \dfrac{1}{3}$

$\dfrac{1}{3} < AP < \dfrac{1}{2}$

Example 19 Prove that for all real values of x,
$-7 \le \sin^2 x - 24\sin x\cos x + 11\cos^2 x \le 19$.



$\sin^2 x - 24\sin x\cos x + 11\cos^2 x$

$= \dfrac{1-\cos 2x}{2} - 12\sin 2x + 11\left(\dfrac{1+\cos 2x}{2}\right)$

$= 6 + 5\cos 2x - 12\sin 2x$

$= 6 + 13\left(\cos 2x \cdot \dfrac{5}{13} - \sin 2x \cdot \dfrac{12}{13}\right)$

$= 6 + 13(\cos 2x\cos\alpha - \sin 2x\sin\alpha)$

$= 6 + 13\cos(2x + \alpha)$

$-1 \le \cos(2x + \alpha) \le 1$

$-13 \le 13\cos(2x + \alpha) \le 13$

$-7 \le 6 + 13\cos(2x + \alpha) \le 19$

$-7 \le \sin^2 x - 24\sin x\cos x + 11\cos^2 x \le 19$.

Half angle formulae

$\cos 2A = 2\cos^2 A - 1 \cdots (11)$

$\cos 2A = 1 - 2\sin^2 A \cdots (12)$

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A} \cdots (13)$

Formulae (11) and (12) can be changed into:

$\cos^2 A = \dfrac{1+\cos 2A}{2} \quad \cdots (15)$ or $\cos A = \pm\sqrt{\dfrac{1+\cos 2A}{2}}$ or $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$

$\sin^2 A = \dfrac{1-\cos 2A}{2} \quad \cdots (16)$ or $\sin A = \pm\sqrt{\dfrac{1-\cos 2A}{2}}$ or $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \quad \cdots (17)$$

Using formulae (15), (16) and (17), we can find the trigonometric ratios of 22.5°, 67.5°, 112.5°, 157.5°, 202.5°, 247.5°,292.5°,337.5°. Equivalently, in radians: $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$.

$$\sin 22.5^\circ = \sqrt{\frac{1-\cos 2\times 22.5^\circ}{2}} = \sqrt{\frac{1-\cos 45^\circ}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\cos 22.5^\circ = \sqrt{\frac{1+\cos 2\times 22.5^\circ}{2}} = \sqrt{\frac{1+\cos 45^\circ}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$1 = \tan(2\times 22.5^\circ) = \dfrac{2\tan 22.5^\circ}{1-\tan^2 22.5^\circ}$

Cross multiplying: $1 - \tan^2 22.5^\circ = 2\tan 22.5^\circ$

$\tan^2 22.5^\circ + 2\tan 22.5^\circ - 1 = 0$

This is a quadratic equation in tan 22.5°.

$\tan 22.5^\circ = -1 \pm \sqrt{2}$

$\because\ 0 < \tan 22.5^\circ < \tan 45^\circ = 1$

$\therefore\ \tan 22.5^\circ = -1 + \sqrt{2}$

$\sin 292.5^\circ = \sin(270^\circ + 22.5^\circ) = -\cos 22.5^\circ = -\dfrac{\sqrt{2+\sqrt{2}}}{2}$

$\cos 292.5^\circ = \cos(270^\circ + 22.5^\circ) = \sin 22.5^\circ = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

$\tan 292.5^\circ = \tan(270^\circ + 22.5^\circ) = -\cot 22.5^\circ = -\dfrac{1}{-1+\sqrt{2}} = -\left(1+\sqrt{2}\right)$

The other trigonometric ratios are similarly found.

Triple angle formulae (三倍角公式)

$\sin 3A = \sin(2A + A)$
$= \sin 2A \cos A + \cos 2A \sin A$
$= 2 \sin A \cos A \cos A + (1 - 2 \sin^2 A) \sin A$
$= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A$
$= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A$

$\sin 3A = 3 \sin A - 4 \sin^3 A \cdots (18)$ $\quad$ $\sin 3A = 3\times? - 4\times??$ Express $\sin 3A$ in terms of $\sin A$ only

$\cos 3A = \cos (2A + A)$
$= \cos 2A \cos A - \sin 2A \sin A$
$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \sin A$
$= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A)$
$= 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A$

$\cos 3A = 4 \cos^3 A - 3 \cos A \cdots (19)$ $\quad$ Express $\cos 3A$ in terms of $\cos A$ only

$$\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$= \frac{\left(\frac{2\tan A}{1-\tan^2 A} + \tan A\right)}{\left(1 - \frac{2\tan A}{1-\tan^2 A}\cdot \tan A\right)} \cdot \frac{(1-\tan^2 A)}{(1-\tan^2 A)}$$

$$= \frac{2\tan A + \tan A(1-\tan^2 A)}{(1-\tan^2 A) - 2\tan^2 A}$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \cdots (20)$$ Express $\tan 3A$ in terms of $\tan A$ only.

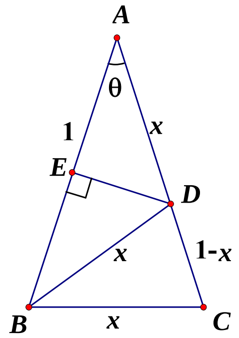
Classwork 4

Prove that $\tan 4A = \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A} \cdots (21)$

and $\tan 5A = \dfrac{5\tan A - 10\tan^3 A + \tan^5 A}{1 - 10\tan^2 A + 5\tan^4 A}. \cdots (22)$

Example 20 Without using calculators, express sin 18°, cos 36° and tan 18° in surd form.
Let $\theta = 18°$, then $5\theta = 90°$, $3\theta = 90° - 2\theta$
$\cos 3\theta = \cos(90° - 2\theta) = \sin 2\theta$
$4\cos^3\theta - 3\cos\theta = 2\sin\theta\cos\theta$
$\because \cos\theta = \cos 18° \neq 0$, divide both sides by $\cos\theta$
$4\cos^2\theta - 3 = 2\sin\theta$
$4(1 - \sin^2\theta) - 3 = 2\sin\theta$
$4 - 4\sin^2\theta - 3 = 2\sin\theta$
$4\sin^2\theta + 2\sin\theta - 1 = 0$
This is a quadratic equation in $\sin\theta$.
$$\sin 18° = \frac{-1 \pm \sqrt{5}}{4}$$
$\because 0° < \theta < 90° \therefore \sin 18° > 0$
$$\sin 18° = \frac{-1 + \sqrt{5}}{4}$$
Let $\theta = 36°$, then $5\theta = 180°$, $3\theta = 180° - 2\theta$
$\sin 3\theta = \sin(180° - 2\theta) = \sin 2\theta$
$3\sin\theta - 4\sin^3\theta = 2\sin\theta\cos\theta$
$\because \sin\theta = \sin 36° \neq 0$, divide both sides by $\sin\theta$
$3 - 4\sin^2\theta = 2\cos\theta$
$3 - 4(1 - \cos^2\theta) = 2\cos\theta$
$3 - 4 + 4\cos^2\theta = 2\cos\theta$
$4\cos^2\theta - 2\cos\theta - 1 = 0$
This is a quadratic equation in $\cos\theta$.
$$\cos 36° = \frac{1 \pm \sqrt{5}}{4}$$
$\because 0° < \theta < 90° \therefore \cos 36° > 0$
$$\cos 36° = \frac{1 + \sqrt{5}}{4}$$
Let $\beta = 18°$, then $5\beta = 90°$, $3\beta = 90° - 2\beta$, let $\tan\beta = t$
$\tan 3\beta = \tan(90° - 2\beta) = \cot 2\beta$
$$\frac{3t - t^3}{1 - 3t^2} = \frac{1 - t^2}{2t}$$
$6t^2 - 2t^4 = 1 - t^2 - 3t^2 + 3t^4$
$5t^4 - 10t^2 + 1 = 0$
This is a quadratic equation in t^2.
$$t^2 = \frac{5 \pm \sqrt{20}}{5} \Rightarrow t = \sqrt{\frac{5 \pm \sqrt{20}}{5}}$$
$\because 0° < 18° < 45° \therefore 0 < \tan 18° < \tan 45° = 1$ and $\sqrt{\frac{5 + \sqrt{20}}{5}} > 1$
$$t^2 = \frac{5 - \sqrt{20}}{5} = \frac{5 - 2\sqrt{5}}{5}$$
$$\tan 18° = \sqrt{\frac{5 - 2\sqrt{5}}{5}}$$

Example 21 Find cos 36° without using triple angle formula.
Consider the following triangle ΔABC.
Given $AB = AC = 1$. D is a point lying on AC such that $AD = BD = BC = x$.



Let $\angle A = \theta$, $CD = 1 - x$.
Then $\angle ABD = \theta$, (base $\angle$s isos. Δ)
$\angle BDC = 2\theta$ (ext. $\angle$ of Δ)
$\angle ACB = 2\theta$ (base $\angle$s isos. Δ)
$\angle ABC = 2\theta$ (base $\angle$s isos. Δ)
$\angle CBD = 2\theta - \theta = \theta$
$\Delta ABC \sim \Delta BCD$ (equiangular)
In ΔABC, $\theta + 2\theta + 2\theta = 180°$ ($\angle$ sum of Δ)
$\theta = 36°$

$\frac{AB}{BC} = \frac{BC}{CD}$ (corr. sides, ~Δs)

$\frac{1}{x} = \frac{x}{1-x}$

$1 - x = x^2$

$x^2 + x - 1 = 0$

$x = \frac{-1+\sqrt{5}}{2}$ or $\frac{-1-\sqrt{5}}{2}$ (< 0, rejected)

Draw $DE \perp AB$ as shown. Then $\Delta ADE \cong \Delta BDE$ (R.H.S.)

$AE = ED = \frac{1}{2}$ (corr. sides, ≅ Δs)

$$\cos 36° = \frac{AE}{AD} = \frac{\frac{1}{2}}{x} = \frac{\frac{1}{2}}{\frac{-1+\sqrt{5}}{2}} = \frac{1}{-1+\sqrt{5}} = \frac{1}{-1+\sqrt{5}} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{1+\sqrt{5}}{4}$$

Example 22 Solve the cubic equation $x^3 - 3x - 1 = 0$ and hence prove that cos 20° is irrational.
Let $x = 2\cos\theta$, then the equation becomes $(2\cos\theta)^3 - 3(2\cos\theta) - 1 = 0$

$8\cos^3\theta - 6\cos\theta = 1$

$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

$\cos 3\theta = \frac{1}{2}$

$3\theta = 60°, 300°, 420°, 660°, 780°, 1020°$
$\theta = 20°, 100°, 140°, 220°, 260°, 340°$
$x = 2\cos\theta = 2\cos 20°, 2\cos 100°, 2\cos 140°, 2\cos 220°, 2\cos 260°, 2\cos 340°$
$\because \cos 340° = \cos 20°, \cos 260° = \cos 100°, \cos 220° = \cos 140°$
$\therefore$ The 3 roots of $x^3 - 3x - 1 = 0$ are $2\cos 20°, 2\cos 100°, 2\cos 140° = 1.879, -1.532, -0.347$.
To prove cos 20° is irrational, we use the method of contradiction.
If cos 20° is rational, then 2 cos 20° is also rational $\therefore$ One root of the cubic equation is rational

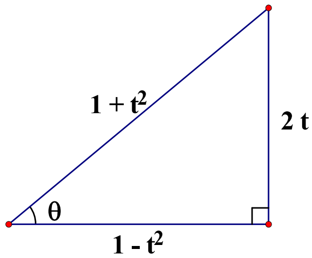
Suppose $2\cos 20° = \frac{b}{a}$, then by factor theorem $ax - b$ is a factor of $x^3 - 3x - 1$

$a = 1, b = \pm 1$
Let $f(x) = x^3 - 3x - 1$
$f(1) = 1^3 - 3 - 1 \ne 0$ and $f(-1) = -1 + 3 - 1 \ne 0$
$\therefore x^3 - 3x - 1 = 0$ has no rational root
Our supposition is false. i.e. cos 20° is irrational.

Circular function of $t = \tan\frac{\theta}{2}$.

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot\cos^2\frac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2t}{1+t^2} \quad \cdots (23)$$

$$\tan\theta = \tan 2\left(\frac{\theta}{2}\right) = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{2t}{1-t^2} \quad \cdots (24)$$



$$\cos\theta = \frac{\sin\theta}{\tan\theta} = \frac{2t}{1+t^2}\times\frac{1-t^2}{2t} = \frac{1-t^2}{1+t^2} \quad \cdots (25)$$

Example 23 Solve 2 sin θ – 3 cos θ = 1 by using circular function of $t = \tan\frac{\theta}{2}$.

$$\frac{2(2t)}{1+t^2} - \frac{3(1-t^2)}{1+t^2} = 1$$

$4t - 3 + 3t^2 = 1 + t^2$

$2t^2 + 4t - 4 = 0$

$t^2 + 2t - 2 = 0$

$t = \tan\frac{\theta}{2} = -1 \pm \sqrt{3}$

$\frac{\theta}{2}$ = 36.206° or –69.896° (rejected) or 216.206° (rejected) or 110.104°

θ = 72.4° or 220.2° (correct to 1 decimal place)

$$\frac{\sin x}{1+\cos x} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2} \cdots (26)$$

$$\frac{\sin x}{1+\cos x} = \frac{\sin x}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x} = \frac{\sin x(1-\cos x)}{1-\cos^2 x} = \frac{1-\cos x}{\sin x} \cdots (27)$$

$$\therefore\ \tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

Example 24 If $\sin x = -\frac{3}{5}$ and 270° < x < 360°, find $\tan\frac{x}{2}$.

$$\cos x = \sqrt{1-\sin^2 x} = \frac{4}{5}$$

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{-\frac{3}{5}}{1+\frac{4}{5}} = -\frac{1}{3}$$

Example 25 Given θ is in third quadrant and $\tan\theta = \frac{4}{3}$, evaluate $\tan\frac{\theta}{2}$ without using calculators.

$\frac{4}{3} = \frac{2t}{1-t^2}$, where $t = \tan\frac{\theta}{2}$, 180° < θ < 270°, 90° < $\frac{\theta}{2}$ < 135°, $\tan\frac{\theta}{2} < 0$

$2t^2 + 3t - 2 = 0$

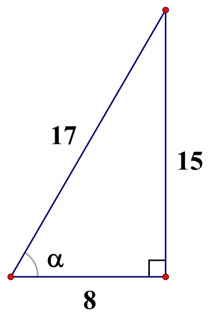
$t = \frac{1}{2}$ (rejected, ∵ 90° < $\frac{\theta}{2}$ < 135°, $\tan\frac{\theta}{2} < 0$) or –2 ⇒ $\tan\frac{\theta}{2} = -2$

Example 26 Given $8\sin\theta + 15\cos\theta = -15$, where $0° \le \theta \le 360°$.

(a) Use subsidiary angle to solve the equation.

(b) Use circular function of $t = \tan\dfrac{\theta}{2}$ to solve the equation.

(c) Which method is correct? Explain briefly.



(a) $\sqrt{8^2+15^2} = 17$, $\cos\alpha = \dfrac{8}{17}$, $\sin\alpha = \dfrac{15}{17}$, $\alpha = 61.9275°$

$$17\cdot\left(\sin\theta\cdot\frac{8}{17}+\cos\theta\cdot\frac{15}{17}\right)=-15$$

$$\sin\theta\cos\alpha + \cos\theta\sin\alpha = -\frac{15}{17}$$

$$\sin(\theta+\alpha) = -\frac{15}{17}$$

$\theta + 61.9275° = -61.9275°, 241.9275°$ or $298.0725°$

$\theta = -123.8°$ (rejected), $180°$ or $236.1450°$

(b) Let $t = \tan\dfrac{\theta}{2}$, $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$.

Then the equation $8\sin\theta + 15\cos\theta = -15$ becomes

$$\frac{8\cdot 2t}{1+t^2}+\frac{15\left(1-t^2\right)}{1+t^2}=-15$$

$$16t + 15 - 15t^2 = -15 - 15t^2$$

$$t = \tan\frac{\theta}{2} = -\frac{15}{8}$$

$$\frac{\theta}{2} = 118.0725° \text{ or } 298.0725°$$

$\theta = 236.1450°$ only

(c) Note that $t = \tan\dfrac{\theta}{2}$ is undefined for $\dfrac{\theta}{2} = 90°$

i.e. $\theta = 180°$ cannot be found in method 2.

Hence method 1 is correct.

Half angle formula in terms of the sides of a triangle (outside syllabus)

Given a triangle ABC. Let $BC = a$, $CA = b$, $AB = c$.

Let $s = \frac{1}{2}(a+b+c)$, $2s = a + b + c$, $2s - 2a = b + c - a$, $2s - 2b = c + a - b$, $2s - 2c = a + b - c$

By Heron's formula, area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$. $\cdots$ (28)

On the other hand, area of triangle $= \frac{1}{2} bc \sin A$

$$\frac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\therefore \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} \quad \cdots (29)$$

Cosine formula gives $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$$\therefore 2\cos^2\frac{A}{2} - 1 = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos^2\frac{A}{2} = \frac{b^2+c^2-a^2+2bc}{4bc}$$

$$\cos\frac{A}{2} = \pm\frac{1}{2}\sqrt{\frac{(b+c)^2-a^2}{bc}} = \pm\frac{1}{2}\sqrt{\frac{(b+c+a)(b+c-a)}{bc}} = \pm\sqrt{\frac{s(s-a)}{bc}}$$

For $0° < A < 180°$, $0° < \frac{A}{2} < 90°$, $\cos\frac{A}{2} > 0$

$$\therefore \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \quad \cdots\cdots (30)$$

$$\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

By formula (29) and (30), $\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = 2\sin\frac{A}{2}\sqrt{\frac{s(s-a)}{bc}}$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \cdots\cdots (31)$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{bc}{s(s-a)}} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \cdots\cdots (32)$$

The formulae for $\sin\frac{B}{2}$, $\sin\frac{C}{2}$, $\cos\frac{B}{2}$, $\cos\frac{C}{2}$, $\tan\frac{B}{2}$ and $\tan\frac{C}{2}$ are symmetric.

Example 27

Given θ is an acute angle. If, in ΔABC, $(b + c)\cos\theta = 2\sqrt{bc}\cos\frac{A}{2}$, prove that $a = (b + c)\sin\theta$.

Hence, find a, given $b = 18.7$, $c = 16.4$ and $A = 57°$.

$$(b+c)\cos\theta = 2\sqrt{bc}\cos\frac{A}{2}$$
$$= 2\sqrt{bc}\cdot\sqrt{\frac{s(s-a)}{bc}}$$
$$= 2\sqrt{s(s-a)}$$
$$= \sqrt{(a+b+c)(b+c-a)}$$

$$\cos\theta = \sqrt{\left(\frac{a+b+c}{b+c}\right)\left(\frac{b+c-a}{b+c}\right)}$$
$$= \sqrt{\left(1+\frac{a}{b+c}\right)\left(1-\frac{a}{b+c}\right)}$$
$$= \sqrt{1-\left(\frac{a}{b+c}\right)^2}$$

$$\cos^2\theta = 1-\left(\frac{a}{b+c}\right)^2$$

$$\sin^2\theta = 1 - \cos^2\theta = \left(\frac{a}{b+c}\right)^2$$

$\sin\theta = \frac{a}{b+c}$ ($\because$ θ is acute, $\sin\theta > 0$)

$\therefore$ $a = (b + c)\sin\theta$

$b = 18.7$, $c = 16.4$, $A = 57°$

$$\cos\theta = \frac{2\sqrt{bc}\cos\frac{A}{2}}{b+c}$$
$$= \frac{2\sqrt{18.7\times16.4}\cos\frac{57^\circ}{2}}{18.7+16.4}$$
$$= 0.876928352$$

$\theta = 28.72597648°$

$a = (b + c)\sin\theta = (18.7 + 16.4)\sin 28.72597648° = 16.87$ (correct to4 sig. fig.)

Example 28 In ΔABC, prove that $a^2-(b-c)^2\cos^2\frac{A}{2}=(b+c)^2\sin^2\frac{A}{2}$.

$$\begin{aligned}
\text{L.H.S.} &= a^2-(b-c)^2\cos^2\frac{A}{2}\\
&= a^2-(b-c)^2\cdot\frac{s(s-a)}{bc}\\
&= \frac{a^2bc-(b-c)^2\cdot\frac{1}{4}(a+b+c)(b+c-a)}{bc}\\
&= \frac{a^2bc-(b-c)^2\cdot\frac{1}{4}\left[(b+c)^2-a^2\right]}{bc}\\
&= \frac{a^2bc+\frac{1}{4}a^2\left(b^2-2bc+c^2\right)-\frac{1}{4}(b-c)^2(b+c)^2}{bc}\\
&= \frac{\frac{1}{4}a^2\left(b^2+2bc+c^2\right)-\frac{1}{4}(b-c)^2(b+c)^2}{bc}\\
&= \frac{\frac{1}{4}(b+c)^2\left[a^2-(b-c)^2\right]}{bc}\\
&= \frac{\frac{1}{4}(b+c)^2(a-b+c)(a+b-c)}{bc}\\
&= \frac{(b+c)^2(s-b)(s-c)}{bc}\\
&= (b+c)^2\sin^2\frac{A}{2} = \text{R.H.S.}
\end{aligned}$$

Sum and Product formulae

sin ($A + B$) = sin A cos B + cos A sin B
sin ($A - B$) = sin A cos B – cos A sin B
sin ($A + B$) + sin ($A - B$) = 2 sin A cos B

$\therefore \sin A \cos B = \frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$ ······ (33) Prduct to sum formula

sin ($A + B$) – sin ($A - B$) = 2 cos A sin B

$\therefore \cos A \sin B = \frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$ ······ (34)

cos ($A + B$) = cos A cos B – sin A sin B
cos ($A - B$) = cos A cos B + sin A sin B
cos ($A + B$) + cos ($A - B$) = 2 cos A cos B

$\therefore \cos A \cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right]$ ······ (35)

cos ($A + B$) – cos ($A - B$) = –2 sin A sin B

$\therefore \sin A \sin B = -\frac{1}{2}\left[\cos(A+B)-\cos(A-B)\right]$ ······ (36)

Formulae (33) to (36) are called product to sum formulae.

Let $x = A + B$, $y = A - B$, then $A = \frac{x+y}{2}$, $B = \frac{x-y}{2}$,

(33) becomes $\sin\frac{x+y}{2}\cos\frac{x-y}{2} = \frac{1}{2}(\sin x + \sin y)$

$\therefore \sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$ ······ (37)

(34) becomes $\cos\frac{x+y}{2}\sin\frac{x-y}{2} = \frac{1}{2}(\sin x - \sin y)$

$\therefore \sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$ ······ (38)

(35) becomes $\cos\frac{x+y}{2}\cos\frac{x-y}{2} = \frac{1}{2}(\cos x + \cos y)$

$\therefore \cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$ ······ (39)

(36) becomes $\sin\frac{x+y}{2}\sin\frac{x-y}{2} = -\frac{1}{2}(\cos x - \cos y)$

$\therefore \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$ ······ (40)

Formulae (37) to (40) are called sum to product formulae.

Memorize: S + S $\rightarrow$ 2SC
S – S $\rightarrow$ 2CS
C + C $\rightarrow$ 2CC
C – C $\rightarrow$ –2SS

There are no formulae for S + C, S – C, T + T or T – T !

Example 29 If cos 16° = sin 14° + sin d° and 0 < d < 90, find d without using calculators.

sin d° = cos 16° – sin 14° = sin 74° – sin 14°

sin d° = $2\cos\frac{74^\circ+14^\circ}{2}\sin\frac{74^\circ-14^\circ}{2}$

sin d° = cos 44° = sin 46°

$d = 46$

Example 30

Express $\dfrac{\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ}$ in the form $a + b\sqrt{c}$, where a, b and c are integers.

$\cos n^\circ + \sin n^\circ = \sqrt{2}\left(\cos 45^\circ \cos n^\circ + \sin 45^\circ \sin n^\circ\right)$ (Subsidiary angles)

$= \sqrt{2}\cos\left(45^\circ - n^\circ\right)$

$(\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ) + (\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ)$

$= \sqrt{2}\left(\cos 44^\circ + \cos 43^\circ + \cdots + \cos 1^\circ\right)$

$\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ = \left(\sqrt{2} - 1\right)\left(\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ\right)$

$$\frac{\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ} = \frac{1}{\sqrt{2} - 1} = 1 + \sqrt{2}$$

Method 2 Recall $\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$ and $\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$.

Take $A = 45^\circ - n^\circ$ and $B = n^\circ$ for $n = 1, 2, \cdots, 22$, we have

$$\frac{\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ}{\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ} = \frac{2\cos\frac{45^\circ}{2}\left(\cos\frac{43^\circ}{2} + \cos\frac{41^\circ}{2} + \cdots + \cos\frac{1^\circ}{2}\right)}{2\sin\frac{45^\circ}{2}\left(\cos\frac{43^\circ}{2} + \cos\frac{41^\circ}{2} + \cdots + \cos\frac{1^\circ}{2}\right)}$$

$= \cot 22.5^\circ = \dfrac{1}{\tan 22.5^\circ} = \dfrac{1}{\sqrt{2} - 1}$ (found on page 14)

$= 1 + \sqrt{2}$

Example 31 Find the value of $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$ without using calculators.

$\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

$= (\sin 80^\circ \sin 20^\circ)(\sin 40^\circ)(\sin 60^\circ)$

$= -\dfrac{1}{2}\left(\cos 100^\circ - \cos 60^\circ\right)\cdot\left(\sin 40^\circ\right)\left(\dfrac{\sqrt{3}}{2}\right)$ ($\because \sin 60^\circ = \dfrac{\sqrt{3}}{2}$)

$= -\dfrac{\sqrt{3}}{4}\left(\cos 100^\circ \sin 40^\circ - \dfrac{1}{2}\sin 40^\circ\right)$ ($\because \cos 60^\circ = \dfrac{1}{2}$)

$= -\dfrac{\sqrt{3}}{4}\left[\dfrac{1}{2}\left(\sin 140^\circ - \sin 60^\circ\right) - \dfrac{1}{2}\sin 40^\circ\right]$

$= -\dfrac{\sqrt{3}}{8}\left(\sin 140^\circ - \sin 60^\circ - \sin 40^\circ\right)$

$= -\dfrac{\sqrt{3}}{8}\left(\sin 40^\circ - \dfrac{\sqrt{3}}{2} - \sin 40^\circ\right)$

$= \dfrac{3}{16}$

Classwork 5 Without using calculators, prove that

(a) $\sin 23^\circ \sin 40^\circ - \cos 43^\circ \cos 74^\circ + \cos 24^\circ \sin 83^\circ = \cos 17^\circ$;

(b) $\cos 17^\circ \cos 81^\circ + \sin 43^\circ \cos 69^\circ + \dfrac{1}{2}\cos 142^\circ = 0$.

Example 32 Without using calculators, find the value of $\sin 12° \sin 24° \sin 48° \sin 96°$.

$\sin 12° \sin 24° \sin 48° \sin 96°$

$= (\sin 12° \sin 48°) (\sin 24° \sin 96°)$

$$= \frac{1}{4}(\cos 36^\circ - \cos 60^\circ)(\cos 72^\circ - \cos 120^\circ)$$

$$= \frac{1}{4}\left(\cos 36^\circ - \frac{1}{2}\right)\left(\cos 72^\circ + \frac{1}{2}\right)$$

$$= \frac{1}{16}(4\cos 36^\circ \cos 72^\circ + 2\cos 36^\circ - 2\cos 72^\circ - 1)$$

$$= \frac{1}{16}\left(1 + \frac{2\sin 36^\circ \cos 36^\circ - 2\sin 36^\circ \cos 72^\circ}{\sin 36^\circ} - 1\right)$$

$$= \frac{1}{16}\left(\frac{\sin 72^\circ - \sin 108^\circ + \sin 36^\circ}{\sin 36^\circ}\right) = \frac{1}{16}$$

Example 33 In ΔABC, prove that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$.

$$\text{LHS} = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= -\frac{1}{2}\left(\cos\frac{A+B}{2} - \cos\frac{A-B}{2}\right)\sin\frac{C}{2}$$

$$= \frac{1}{2}\left(\cos\frac{A-B}{2} - \sin\frac{C}{2}\right)\sin\frac{C}{2}$$

$$\le \frac{1}{2}\left(1 - \sin\frac{C}{2}\right)\sin\frac{C}{2} \quad \text{(equality holds when } A = B\text{)}$$

$$\le \frac{1}{2}\left(\frac{1 - \sin\frac{C}{2} + \sin\frac{C}{2}}{2}\right)^2 \quad \text{(G.M. } \le \text{ A.M.: } ab \le \left(\frac{a+b}{2}\right)^2 \text{ for } a \ge 0, b \ge 0\text{)}$$

$$= \frac{1}{8} = \text{R.H.S. (equality holds when } 1 - \sin\frac{C}{2} = \sin\frac{C}{2}\text{, i.e. } C = \frac{\pi}{3}\text{)}$$

Example 34 If $A + B + C = 180°$, prove that $\cos^2 A + (\cos B + \cos C)^2 - 1 = 4\sin^2\frac{A}{2}\cos B\cos C$

Hence, if also $\cos B + \cos C - \cos A = 1$, prove that $\sec A - \sec B - \sec C = 1$.

L.H.S. $= \cos^2 A + (\cos B + \cos C)^2 - 1 = \cos^2 A + \left(2\cos\frac{B+C}{2}\cos\frac{B-C}{2}\right)^2 - 1$

$$= \cos^2 A + 4\left(\cos\frac{180^\circ - A}{2}\cos\frac{B-C}{2}\right)^2 - 1 = 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2} - \sin^2 A$$

$$= 4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2} - \left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)^2 = 4\sin^2\frac{A}{2}\left(\cos^2\frac{B-C}{2} - \cos^2\frac{A}{2}\right)$$

$$= 4\sin^2\frac{A}{2}\left[\cos^2\frac{B-C}{2} - \cos\frac{180^\circ - (B+C)}{2}\right] = 4\sin^2\frac{A}{2}\left(\cos^2\frac{B-C}{2} - \sin^2\frac{B+C}{2}\right)$$

$$= 4\sin^2\frac{A}{2}\left[\frac{1+\cos(B+C)}{2} - \frac{1-\cos(B+C)}{2}\right] = 2\sin^2\frac{A}{2}\left[\cos(B+C) + \cos(B-C)\right]$$

$$= 4\sin^2\frac{A}{2}\cos B\cos C = \text{R.H.S.}$$

If also $\cos B + \cos C - \cos A = 1$, then $\cos B + \cos C = 1 + \cos A \cdots\cdots (*)$

$\cos^2 A + (\cos B + \cos C)^2 - 1 = 4\sin^2\frac{A}{2}\cos B\cos C$ (proved)

$\cos^2 A + (1 + \cos A)^2 - 1 = 4\sin^2\frac{A}{2}\cos B\cos C$ by (*)

$2\cos^2 A + 2\cos A = 4\left(\frac{1-\cos A}{2}\right)\cos B\cos C \Rightarrow \cos A(\cos A + 1) = (1 - \cos A)\cos B\cos C$

$\cos A(\cos B + \cos C) = (1 - \cos A)\cos B\cos C$ by (*)

$\cos B\cos C - \cos C\cos A - \cos A\cos B = \cos A\cos B\cos C$

$$\frac{\cos B\cos C}{\cos A\cos B\cos C} - \frac{\cos C\cos A}{\cos A\cos B\cos C} - \frac{\cos A\cos B}{\cos A\cos B\cos C} = \frac{\cos A + \cos B\cos C}{\cos A + \cos B\cos C}$$

$\sec A - \sec B - \sec C = 1$

Example 35 Prove that $\frac{1}{2\sin\theta}\left(\operatorname{cosec}2\theta-\operatorname{cosec}4\theta\right)=\frac{\cos 3\theta}{\sin 2\theta\sin 4\theta}$.

Hence find the sum to *n* terms of the series $\frac{\cos 3\theta}{\sin 2\theta\sin 4\theta}+\frac{\cos 5\theta}{\sin 4\theta\sin 6\theta}+\frac{\cos 7\theta}{\sin 6\theta\sin 8\theta}+\cdots$.

$$\begin{aligned}
\text{L.H.S.} &= \frac{1}{2\sin\theta}\left(\operatorname{cosec}2\theta-\operatorname{cosec}4\theta\right)\\
&= \frac{1}{2\sin\theta}\left(\frac{1}{\sin 2\theta}-\frac{1}{\sin 4\theta}\right)\\
&= \frac{\sin 4\theta-\sin 2\theta}{2\sin\theta\sin 2\theta\sin 4\theta}\\
&= \frac{2\sin 2\theta\cos 2\theta-\sin 2\theta}{2\sin\theta\sin 2\theta\sin 4\theta}\\
&= \frac{\sin 2\theta\left(2\cos 2\theta-1\right)}{2\sin\theta\sin 2\theta\sin 4\theta}\\
&= \frac{2\sin\theta\cos\theta\left[2\left(2\cos^2\theta-1\right)-1\right]}{2\sin\theta\sin 2\theta\sin 4\theta}\\
&= \frac{4\cos^3\theta-3\cos\theta}{\sin 2\theta\sin 4\theta}\\
&= \frac{\cos 3\theta}{\sin 2\theta\sin 4\theta}=\text{R.H.S.}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2\sin\theta}\left(\operatorname{cosec}2r\theta-\operatorname{cosec}\left(2r+2\right)\theta\right) &= \frac{1}{2\sin\theta}\left(\frac{1}{\sin 2r\theta}-\frac{1}{\sin 2\left(r+1\right)\theta}\right)\\
&= \frac{\sin\left(2r\theta+2\theta\right)-\sin 2r\theta}{2\sin\theta\sin 2r\theta\sin 2\left(r+1\right)\theta}\\
&= \frac{2\cos\left(2r\theta+\theta\right)\sin\theta}{2\sin\theta\sin 2r\theta\sin 2\left(r+1\right)\theta}\\
&= \frac{\cos 2\left(r+1\right)\theta}{\sin 2r\theta\sin 2\left(r+1\right)\theta}
\end{aligned}$$

$$\frac{\cos 3\theta}{\sin 2\theta\sin 4\theta}+\frac{\cos 5\theta}{\sin 4\theta\sin 6\theta}+\frac{\cos 7\theta}{\sin 6\theta\sin 8\theta}+\cdots+\frac{\cos\left(2n+1\right)\theta}{\sin 2n\theta\sin\left(2n+2\right)\theta}$$

$$=\sum_{r=1}^{n}\frac{\cos\left(2r+1\right)\theta}{\sin 2r\theta\sin\left(2r+2\right)\theta}$$

$$=\sum_{r=1}^{n}\frac{1}{2\sin\theta}\left(\operatorname{cosec}2r\theta-\operatorname{cosec}\left(2r+2\right)\theta\right)$$

$$=\frac{1}{2\sin\theta}\left(\frac{1}{\sin 2\theta}-\frac{1}{\sin\left(2n+2\right)\theta}\right)$$

Example 36 In ΔABC, if $2\sin B = \sin A + \sin C$, then $2\cot\frac{B}{2} = \cot\frac{A}{2} + \cot\frac{C}{2}$.

$2\sin B = \sin A + \sin C$ (Given)

$2\sin B = 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$

$2\sin\frac{B}{2}\cos\frac{B}{2} = \cos\frac{B}{2}\cos\frac{A-C}{2}$ $\quad(\because \frac{A+C}{2} = 90^\circ - \frac{B}{2})$

$2\sin\frac{B}{2} = \cos\frac{A-C}{2}$ $\quad$ (clearly $\cos\frac{B}{2} \neq 0$)

$2\sin\frac{180^\circ-(A+C)}{2} = \cos\frac{A-C}{2}$

$2\cos\frac{A+C}{2} = \cos\frac{A-C}{2}$

$2\cos\frac{A}{2}\cos\frac{C}{2} - 2\sin\frac{A}{2}\sin\frac{C}{2} = \cos\frac{A}{2}\cos\frac{C}{2} + \sin\frac{A}{2}\sin\frac{C}{2}$

$\cos\frac{A}{2}\cos\frac{C}{2} = 3\sin\frac{A}{2}\sin\frac{C}{2}$

$\cot\frac{A}{2}\cot\frac{C}{2} = 3 \Rightarrow \cot\frac{A}{2} = \frac{3}{\cot\frac{C}{2}}$ $\cdots$ (*) or $\tan\frac{A}{2} = \frac{1}{3\tan\frac{C}{2}}$

$$\begin{aligned}
2\cot\frac{B}{2} &= 2\cot\frac{180^\circ-(A+C)}{2} \\
&= 2\tan\frac{A+C}{2} \\
&= 2\cdot\frac{\tan\frac{A}{2}+\tan\frac{C}{2}}{1-\tan\frac{A}{2}\cdot\tan\frac{C}{2}} \\
&= 2\cdot\frac{\frac{1}{3\tan\frac{C}{2}}+\tan\frac{C}{2}}{1-\frac{1}{3\tan\frac{C}{2}}\cdot\tan\frac{C}{2}} \quad \text{by (*)} \\
&= \frac{1}{\tan\frac{C}{2}} + 3\tan\frac{C}{2} \\
&= \cot\frac{C}{2} + \frac{3}{\cot\frac{C}{2}} \\
&= \cot\frac{A}{2} + \cot\frac{C}{2} \quad \text{by (*)}
\end{aligned}$$

Example 37 If $A + B + C = 180°$, prove that

$\sin^3 A \sin (B - C) + \sin^3 B \sin (C - A) + \sin^3 C \sin (A - B) = 0$

But $\sin^2 A \sin (B - C) + \sin^2 B \sin (C - A) + \sin^2 C \sin (A - B) = 0$ only if ΔABC is isosceles.

$$\begin{aligned}
\sin^3 A \sin (B - C) &= \sin^2 A \cdot \frac{1}{2} [\cos (A + C - B) - \cos (A + B - C)] \\
&= \frac{1}{2} \sin^2 A [\cos (180° - 2B) - \cos (180° - 2C)] \\
&= \frac{1}{2} \sin^2 A (\cos 2C - \cos 2B) \\
&= \frac{1}{2} \sin^2 A (1 - 2 \sin^2 C - 1 + 2 \sin^2 B) \\
&= \frac{1}{2} \sin^2 A (2 \sin^2 B - 2 \sin^2 C) \\
&= \sin^2 A (\sin^2 B - \sin^2 C)
\end{aligned}$$

Similarly, $\sin^3 B \sin (C - A) = \sin^2 B (\sin^2 C - \sin^2 A)$, $\sin^3 C \sin (A - B) = \sin^2 C (\sin^2 A - \sin^2 B)$

$\therefore \sin^3 A \sin (B - C) + \sin^3 B \sin (C - A) + \sin^3 C \sin (A - B)$

$= \sin^2 A (\sin^2 B - \sin^2 C) + \sin^2 B (\sin^2 C - \sin^2 A) + \sin^2 C (\sin^2 A - \sin^2 B) = 0$

If $\sin^2 A \sin (B - C) + \sin^2 B \sin (C - A) + \sin^2 C \sin (A - B) = 0$, then

$\sin A (\sin^2 B - \sin^2 C) + \sin B (\sin^2 C - \sin^2 A) + \sin C (\sin^2 A - \sin^2 B) = 0$

$\sin A (\sin B - \sin C) (\sin B + \sin C) + \sin B \sin C (\sin C - \sin B) + \sin^2 A (\sin C - \sin B) = 0$

$(\sin B - \sin C)[\sin A (\sin B + \sin C) - \sin B \sin C - \sin^2 A] = 0$

$(\sin B - \sin C) (\sin A - \sin B) (\sin C - \sin A) = 0$

$A = B$ or $B = C$ or $C = A$

$\therefore \Delta ABC$ is isosceles

Example 38 If n is an odd integer, and A, B, C are the angles of a triangle, prove that

$\sin nA + \sin nB + \sin nC = 4\sin\frac{n\pi}{2}\cos\frac{nA}{2}\cos\frac{nB}{2}\cos\frac{nC}{2}$.

L.H.S. = $(\sin nA + \sin nB) + \sin nC$

$$=2\sin\frac{n(A+B)}{2}\cos\frac{n(A-B)}{2}+\sin nC=2\sin\frac{n(\pi-C)}{2}\cos\frac{n(A-B)}{2}+2\sin\frac{nC}{2}\cos\frac{nC}{2}$$

$$=2\sin\left[k\pi+\left(\frac{\pi}{2}-\frac{nC}{2}\right)\right]\cos\frac{n(A-B)}{2}+2\sin\frac{nC}{2}\cos\frac{nC}{2}\text{, where } n=2k+1$$

$$=2(-1)^k\cos\frac{nC}{2}\cos\frac{n(A-B)}{2}+2\sin\frac{nC}{2}\cos\frac{nC}{2}$$

$$=2\sin\left(k\pi+\frac{\pi}{2}\right)\cos\frac{nC}{2}\cos\frac{n(A-B)}{2}+2\sin\frac{n[\pi-(A+B)]}{2}\cos\frac{nC}{2}$$

$$=2\cos\frac{nC}{2}\left\{\sin\frac{n\pi}{2}\cos\frac{n(A-B)}{2}+\sin\left[k\pi+\frac{\pi}{2}-\frac{n(A+B)}{2}\right]\right\}$$

$$=2\cos\frac{nC}{2}\left[\sin\frac{n\pi}{2}\cos\frac{n(A-B)}{2}+(-1)^k\cos\frac{n(A+B)}{2}\right]$$

$$=2\cos\frac{nC}{2}\left[\sin\frac{n\pi}{2}\cos\frac{n(A-B)}{2}+\sin\frac{n\pi}{2}\cos\frac{n(A+B)}{2}\right]$$

$$=2\sin\frac{n\pi}{2}\cos\frac{nC}{2}\left[\cos\frac{n(A+B)}{2}+\cos\frac{n(A-B)}{2}\right]=4\sin\frac{n\pi}{2}\cos\frac{nA}{2}\cos\frac{nB}{2}\cos\frac{nC}{2}=\text{R.H.S.}$$

Remark If the question is changed into: If n is an even integer and A, B, C are the angles of a triangle, find a necessary condition that $\sin nA + \sin nB + \sin nC = 0$, is this condition sufficient?

"Necessity" $\sin nA + \sin nB + \sin nC = 0$

$\Rightarrow \sin 2kA + \sin 2kB + \sin 2kC = 0$, where $n = 2k$

$\Rightarrow 2\sin k(A + B)\cos k(A - B) + 2\sin k[\pi - (A + B)]\cos k[\pi - (A + B)] = 0$

$\Rightarrow \sin k(A + B)\cos k(A - B) + (-1)^{k+1}\sin k(A + B)(-1)^k\cos k(A + B) = 0$

$\Rightarrow \sin k(A + B)[\cos k(A - B) - \cos k(A + B)] = 0$

$\Rightarrow 2\sin k(A + B)\sin kA\sin kB = 0$

$\Rightarrow k(A + B) = m\pi$ or $kA = m\pi$ or $kB = m\pi$

$\therefore$ Necessary condition is: any one angle is $\frac{m\pi}{k}$ or the sum of any two angles is $\frac{m\pi}{k}$.

The condition is "sufficient": If $A=\frac{m\pi}{k}$ or $A + B=\frac{m\pi}{k}$, trace back the "$\Rightarrow$" sign.

Example 39

If $\sin\theta + \sin\phi = a$, $\tan\theta + \tan\phi = b$, $\sec\theta + \sec\phi = c$, prove that $8bc = a[4b^2 + (b^2 - c^2)^2]$.

$b^2 - c^2 = (\tan\theta + \tan\phi)^2 - (\sec\theta + \sec\phi)^2$

$= -2 + 2\tan\theta\tan\phi - 2\sec\theta\sec\phi$, using the identity $\sec^2 A - \tan^2 A = 1$

$$= \frac{-2\cos\theta\cos\phi}{\cos\theta\cos\phi} + \frac{2\sin\theta\sin\phi}{\cos\theta\cos\phi} - \frac{2}{\cos\theta\cos\phi}$$

$$= \frac{-2\cos(\theta+\phi)}{\cos\theta\cos\phi} - \frac{2}{\cos\theta\cos\phi} = -2\cdot\frac{1+\cos(\theta+\phi)}{\cos\theta\cos\phi}$$

$$4b^2 + (b^2 - c^2)^2 = 4(\tan\theta + \tan\phi)^2 + 4\cdot\left[\frac{1+\cos(\theta+\phi)}{\cos\theta\cos\phi}\right]^2$$

$$= 4\cdot\left[\left(\frac{\sin\theta}{\cos\theta} + \frac{\sin\phi}{\cos\phi}\right)^2 + \frac{1+2\cos(\theta+\phi)+\cos^2(\theta+\phi)}{\cos^2\theta\cos^2\phi}\right]$$

$$= 4\cdot\left[\frac{(\sin\theta\cos\phi + \cos\theta\sin\phi)^2}{\cos^2\theta\cos^2\phi} + \frac{1+2\cos(\theta+\phi)+\cos^2(\theta+\phi)}{\cos^2\theta\cos^2\phi}\right]$$

$$= 4\cdot\frac{\sin^2(\theta+\phi)+1+2\cos(\theta+\phi)+\cos^2(\theta+\phi)}{\cos^2\theta\cos^2\phi}$$

$$= 4\cdot\frac{1+1+2\cos(\theta+\phi)}{\cos^2\theta\cos^2\phi}$$

$$= 8\cdot\frac{1+\cos(\theta+\phi)}{\cos^2\theta\cos^2\phi}$$

$$a[4b^2 + (b^2 - c^2)^2] = (\sin\theta + \sin\phi)\cdot 8\cdot\frac{1+\cos(\theta+\phi)}{\cos^2\theta\cos^2\phi}$$

$$= \frac{8}{\cos^2\theta\cos^2\phi}\cdot\left(2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right)\left(2\cos^2\frac{\theta+\phi}{2}\right)$$

$$= \frac{32}{\cos^2\theta\cos^2\phi}\cdot\left(\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right)\left(\cos^2\frac{\theta+\phi}{2}\right)$$

$8bc = 8(\tan\theta + \tan\phi)(\sec\theta + \sec\phi)$

$$= 8\cdot\frac{\sin(\theta+\phi)}{\cos\theta\cos\phi}\cdot\frac{\cos\theta+\cos\phi}{\cos\theta\cos\phi}$$

$$= \frac{8}{\cos^2\theta\cos^2\phi}\cdot\left(2\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2}\right)\cdot\left(2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right)$$

$\therefore\ 8bc = a[4b^2 + (b^2 - c^2)^2]$.

Theorem

Let A, B, C be three angles such that $0 \le A, B, C \le \frac{\pi}{2}$ and $A + B + C = \frac{\pi}{2}$

Then $\sin A + \sin B + \sin C \le \frac{3}{2}$, equality holds when $A = B = C = \frac{\pi}{6}$.

Proof: $C = \frac{\pi}{2} - (A + B)$, $\sin C = \cos (A + B)$

$\sin A + \sin B + \sin C = \sin A + \sin B + \cos (A + B)$

$$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{A+B}{2}$$

$$= 2\sin\frac{A+B}{2}\left(\cos\frac{A-B}{2} - \sin\frac{A+B}{2}\right) + 1$$

$$\le 2\left[\frac{\sin\frac{A+B}{2} + \cos\frac{A-B}{2} - \sin\frac{A+B}{2}}{2}\right]^2 + 1 \quad \because ab \le \left(\frac{a+b}{2}\right)^2$$

equality holds when $a = b$; i.e. $\cos\frac{A-B}{2} = 2\sin\frac{A+B}{2}$ …(1)

$$= \frac{1}{2}\cos^2\frac{A-B}{2} + 1$$

$$\le \frac{1}{2} + 1 = \frac{3}{2}$$, equality holds when $\frac{A-B}{2} = 0$, i.e. $A = B$ ……(2)

Combine (1) and (2), equality holds when $A = B = C = \frac{\pi}{6}$

Example 40 If $0 < x < y < \frac{\pi}{2}$, prove that $\sin x + \cos y - \sin (x - y) \le \frac{\pi}{2}$

Let $A = x$, $B = \frac{\pi}{2} - y$, $C = y - x$, then $0 \le A, B, C \le \frac{\pi}{2}$ and $A + B + C = \frac{\pi}{2}$

$\sin x + \cos y - \sin (x - y) = \sin x + \sin(\frac{\pi}{2} - y) + \sin(y - x)$

$= \sin A + \sin B + \sin C$

$\le \frac{3}{2}$ by the above theorem

$\le \frac{\pi}{2}$

The question is proved.

Example 41 Given $\sin^2 x + \sin^2 y = \sin(x + y)$, where x and y are acute angles. Prove that $x + y = 90°$

Proof: If $x = y$, then the equation becomes $2 \sin^2 x = \sin 2x$

$2 \sin^2 x = 2 \sin x \cos x$

$\because 0 < x < 90° \therefore \sin x \neq 0$

$\sin x = \cos x$

$\tan x = 1 \Rightarrow x = y = 45°$

If $x \neq y$, without loss of generality let $x > y > 0°$.

Since $\sin^2 x + \sin^2 y = (\sin x + \sin y)^2 - 2 \sin x \sin y$

$$= \left(2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\right)^2 - 2\sin x \sin y$$

$$= 4\sin^2\frac{x+y}{2}\cos^2\frac{x-y}{2} + \cos(x+y) - \cos(x-y)$$

$$= [1 - \cos(x + y)][1 + \cos(x - y)] + \cos(x + y) - \cos(x - y)$$

$$= 1 - \cos(x + y)\cos(x - y)$$

Hence $\sin^2 x + \sin^2 y = \sin(x + y)$ becomes

$1 - \cos(x + y)\cos(x - y) = \sin(x + y)$

$\cos(x + y)\cos(x - y) = 1 - \sin(x + y)$ ……….(*)

Suppose $x + y \neq 90°$

If $x + y > 90°$, by (*): LHS < 0, RHS > 0 impossible.

If $x + y < 90°$, since $x > y > 0°$,

then $\cos(x - y) > \cos(x + y) > 0$

By (*) $1 - \sin(x + y) = \cos(x + y) \cos(x - y) > \cos^2(x + y)$

$1 - \sin(x + y) > 1 - \sin^2(x + y)$

$\sin^2(x + y) - \sin(x + y) > 0$

$\sin(x + y)[\sin(x + y) - 1] > 0$

$\sin(x + y) < 0$ or $\sin(x + y) > 1$ which is impossible.

Therefore $x + y = 90°$

The question is proved.

Example 42 If $\sin 5° + \sin 10° + \sin 15° + ... + \sin 170° + \sin 175° = \tan x°$, find x.

$2 (\sin 5° + \sin 10° + \sin 15° + ... + \sin 170° + \sin 175°) \sin 2.5°$

$= \cos 2.5° - \cos 7.5° + \cos 7.5° - \cos 12.5° + \cos 12.5° - \cos 17.5° + ... + \cos 172.5° - \cos 177.5°$

$= \cos 2.5° - \cos 177.5°$

$= 2 \sin 90° \sin 87.5°$

$= 2 \sin 87.5°$

$\sin 5° + \sin 10° + \sin 15° + ... + \sin 170° + \sin 175° = \tan 87.5°$

$x = 87.5$

Example 43

(*a*) Prove that $\sin(\theta + \frac{k-1}{2}\alpha)\sin\frac{k\alpha}{2} + \sin\frac{\alpha}{2}\sin(\theta + k\alpha) = \frac{1}{2}[\cos(\theta - \frac{\alpha}{2}) - \cos(\theta + \frac{2k+1}{2}\alpha)]$

Hence prove that $\sin(\theta + \frac{k-1}{2}\alpha)\sin\frac{k\alpha}{2} + \sin\frac{\alpha}{2}\sin(\theta + k\alpha) = \sin(\theta + \frac{k\alpha}{2})\sin\frac{k+1}{2}\alpha$

(*b*) Prove by mathematical induction that

$$\sin\theta + \sin(\theta + \alpha) + \ldots\ldots + \sin[\theta + (n-1)\alpha] = \frac{\sin\left[\theta + \left(\frac{n-1}{2}\right)\alpha\right]\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

(*a*) $\sin(\theta + \frac{k-1}{2}\alpha)\sin\frac{k\alpha}{2} + \sin\frac{\alpha}{2}\sin(\theta + k\alpha)$

$= -\frac{1}{2}[\cos(\theta + \frac{2k-1}{2}\alpha) - \cos(\theta - \frac{\alpha}{2})] - \frac{1}{2}[\cos(\theta + \frac{2k+1}{2}\alpha) - \cos(\theta + \frac{2k-1}{2}\alpha)]$

$= \frac{1}{2}[\cos(\theta - \frac{\alpha}{2}) - \cos(\theta + \frac{2k+1}{2}\alpha)]$

$\sin(\theta + \frac{k-1}{2}\alpha)\sin\frac{k\alpha}{2} + \sin\frac{\alpha}{2}\sin(\theta + k\alpha)$

$= -\frac{1}{2}[\cos(\theta + \frac{2k+1}{2}\alpha) - \cos(\theta - \frac{\alpha}{2})]$

$= -\frac{1}{2}(-2)[\sin(\theta + \frac{k\alpha}{2})\sin(\theta + \frac{k+1}{2}\alpha)]$

$= \sin(\theta + \frac{k\alpha}{2})\sin\frac{k+1}{2}\alpha$

(*b*) $n = 1$, LHS $= \sin\theta$, RHS $= \frac{\sin\theta\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} = \sin\theta$

It is true for $n = 1$.
Suppose it is true for $n = k$.
i.e. $\sin\theta + \sin(\theta + \alpha) + \ldots\ldots + \sin[\theta + (k-1)\alpha] = \frac{\sin\left[\theta + \left(\frac{k-1}{2}\right)\alpha\right]\sin\frac{k\alpha}{2}}{\sin\frac{\alpha}{2}}$ ....(*)

When $n = k + 1$,
L.H.S. $= \sin\theta + \sin(\theta + \alpha) + \ldots\ldots + \sin[\theta + (k-1)\alpha] + \sin(\theta + k\alpha)$

$= \frac{\sin\left[\theta + \left(\frac{k-1}{2}\right)\alpha\right]\sin\frac{k\alpha}{2}}{\sin\frac{\alpha}{2}} + \sin(\theta + k\alpha)$ by (*)

$= \frac{\sin\left[\theta + \left(\frac{k-1}{2}\right)\alpha\right]\sin\frac{k\alpha}{2} + \sin\frac{\alpha}{2}\sin(\theta + k\alpha)}{\sin\frac{\alpha}{2}}$

$= \frac{\sin\left(\theta + \frac{k\alpha}{2}\right)\sin\frac{k+1}{2}\alpha}{\sin\frac{\alpha}{2}}$ by (*a*)

= R.H.S.

Hence it is also true for $n = k + 1$
By the principle of induction, the statement is true for all positive integer n.

Example 44

In the figure, *ABCDE* is a regular pentagon of side = 1. Prove that $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$.

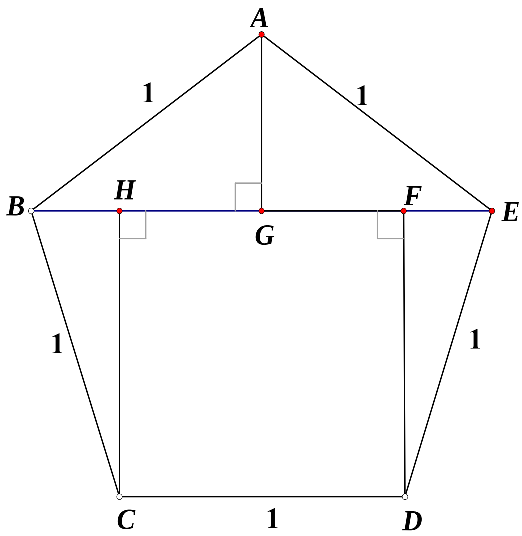
$\angle A = \angle B = \angle C = \angle D = \angle E = \frac{3\pi}{5}$ ($\angle$ sum of polygon)

Join BE. Draw $AG \perp BE$, $CH \perp BE$, $DF \perp BE$.

Then $\Delta ABG \cong \Delta AEG$ (R.H.S.)

$\angle ABG = \angle AEG$ (corr. $\angle$s $\cong$ Δs)

$= \left(\pi - \frac{3\pi}{5}\right) \div 2 = \frac{\pi}{5}$ ($\angle$ sum of isos. Δ)



$\angle CBH = \angle B - \angle ABG = \frac{3\pi}{5} - \frac{\pi}{5} = \frac{2\pi}{5}$

$\angle DEF = \angle E - \angle AEG = \frac{3\pi}{5} - \frac{\pi}{5} = \frac{2\pi}{5}$

$\angle BHC = \frac{\pi}{2} = \angle DFE$ (by construction)

$BC = DE = 1$ (given)

$\Delta BCH \cong \Delta EDF$ (A.A.S.)

$CH = DF$ (corr. sides $\cong$ Δs)

$CH \,//\, DF$ (int. $\angle$s supp.)

CDFH is a rectangle (opp. sides are equal and //)

$BE = BG + GE = 2\cos\frac{\pi}{5}$ $\cdots\cdots$ (1)

$BE = BH + HF + FE = 2\cos\frac{2\pi}{5} + 1$ $\cdots\cdots$ (2)

(1) = (2): $2\cos\frac{\pi}{5} = 2\cos\frac{2\pi}{5} + 1$

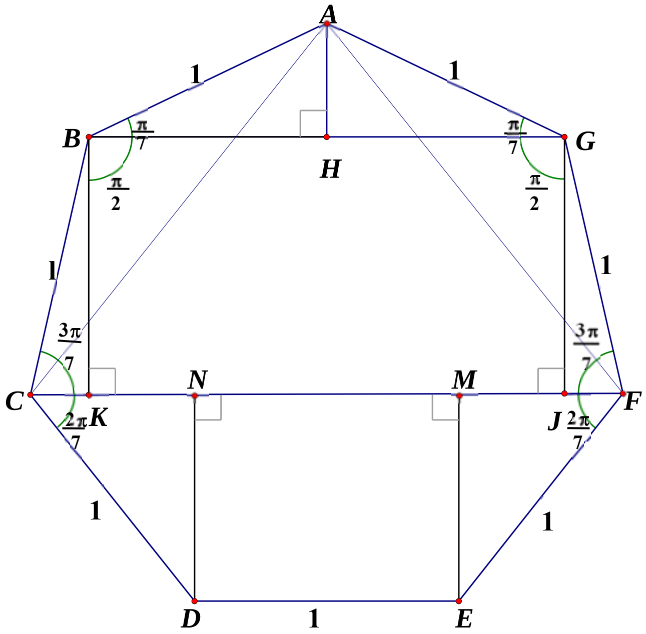
$\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$

$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$

Example 45

In the figure, $ABCDEFG$ is a regular heptagon of side = 1. Prove that $\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$.

$\angle A=\angle B=\angle C=\angle D=\angle E=\angle F=\angle G=\frac{5\pi}{7}$ ($\angle$ sum of polygon)



Join BG, CF. Draw $AH\perp BG$, $BK\perp CF$,
$GJ\perp CF$, $DN\perp CF$, $EM\perp CF$.
Then $\Delta ABH\cong\Delta AGH$ (R.H.S.)
$\angle ABH=\angle AGH$ (corr. $\angle$s $\cong\Delta$s)
$=\left(\pi-\frac{5\pi}{7}\right)\div 2=\frac{\pi}{7}$ ($\angle$ sum of isos. Δ)
Join AC, AF. Then $\Delta ABC\cong\Delta AGF$ (S.A.S.)
$\angle ACB=\angle AFG=\frac{\pi}{7}$ …… (1) (corr. $\angle$s $\cong\Delta$s)
$AC=AF$ (corr. sides $\cong\Delta$s)
$\angle ACF=\angle AFC$ …… (2) (base $\angle$s isos. Δs)
$\therefore\ \angle BCK=\angle GFJ$ by (1) and (2)
$\angle BKC=\frac{\pi}{2}=\angle GJF$ (by construction)
$BC=GF=1$ (given)
$\therefore\ \Delta BCK\cong\Delta GFJ$ (A.A.S.)
$BK=GJ$ (corr. sides $\cong\Delta$s)
$BK\,//\,GJ$ (int. $\angle$s supp.)
$BGJK$ is a rectangle (opp .sides are equal and parallel)
$BG=KJ=2BH=2\cos\frac{\pi}{7}$ (opp. sides of rectangle)

$\angle GBK=\frac{\pi}{2}=\angle BGJ$ (int. $\angle$s $BG\,//\,KJ$)

$\angle CBK=\angle B-\angle GBK-\angle ABH=\frac{5\pi}{7}-\frac{\pi}{2}-\frac{\pi}{7}=\frac{\pi}{7}$

$\angle FGJ=\angle G-\angle BGJ-\angle AGH=\frac{5\pi}{7}-\frac{\pi}{2}-\frac{\pi}{7}=\frac{\pi}{7}$

$\angle BCK=\pi-\frac{\pi}{2}-\frac{\pi}{7}=\frac{3\pi}{7}$ ($\angle$ sum of Δ)

$\angle GFJ=\pi-\frac{\pi}{2}-\frac{\pi}{7}=\frac{3\pi}{7}$ ($\angle$ sum of Δ)

$\angle DCN=\angle C-\angle BCK=\frac{5\pi}{7}-\frac{3\pi}{7}=\frac{2\pi}{7}$

$\angle EFM=\angle F-\angle GFJ=\frac{5\pi}{7}-\frac{3\pi}{7}=\frac{2\pi}{7}$

$CD=EF=1$ (given)

$\angle DNM=\frac{\pi}{2}=\angle EMN$ (by construction)

$\therefore\ \Delta CDN\cong\Delta FEM$ (A.A.S.)
$CN=FM$, $DN=EM$ (corr. sides $\cong\Delta$s)
$DN\,//\,EM$ (int. $\angle$s supp.)
$DEMN$ is a rectangle (opp .sides are equal and parallel)
$MN=1$ (opp. sides of rectangle)

$CF = CN + NM + MF = 2\cos\frac{2\pi}{7}+1$ …… (1)

$CF = CK + KJ + JF = 2\cos\frac{3\pi}{7}+2\cos\frac{\pi}{7}$ …… (2)

(1) = (2): $2\cos\frac{3\pi}{7}+2\cos\frac{\pi}{7}=2\cos\frac{2\pi}{7}+1$

$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}-\cos\frac{2\pi}{7}=\frac{1}{2}$

$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$ ($\because$ $\cos(\pi - \theta) = -\cos\theta$)

Example 46

In general $\cos\frac{\pi}{2n-1}+\cos\frac{3\pi}{2n-1}+\cdots+\cos\frac{(2n-3)\pi}{2n-1}=\frac{1}{2}$ for $n \ge 2$.

Let $y = \cos\frac{\pi}{2n-1}+\cos\frac{3\pi}{2n-1}+\cdots+\cos\frac{(2n-3)\pi}{2n-1}$

$$2y\sin\frac{\pi}{2n-1}=\sin\frac{2\pi}{2n-1}+\left(\sin\frac{4\pi}{2n-1}-\sin\frac{2\pi}{2n-1}\right)+\cdots+\left[\sin\frac{(2n-2)\pi}{2n-1}-\sin\frac{(2n-4)\pi}{2n-1}\right]$$

$$=\sin\frac{(2n-2)\pi}{2n-1}$$

$$=\sin\left(\pi-\frac{\pi}{2n-1}\right)$$

$$=\sin\frac{\pi}{2n-1}$$

$\because 0<\frac{\pi}{2n-1}<\frac{\pi}{2}$

$\therefore \sin\frac{\pi}{2n-1}\ne 0$

$y = \cos\frac{\pi}{2n-1}+\cos\frac{3\pi}{2n-1}+\cdots+\cos\frac{(2n-3)\pi}{2n-1}=\frac{1}{2}$

In particular, $n = 3$, $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$

$n = 4$, $\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$

Find the value of $\cos 1° + \cos 2° + \cos 3° + \cdots + \cos 90°$.

$y = \cos 1° + \cos 2° + \cos 3° + \cdots + \cos 90°$

$2y \sin 0.5° = (\sin 1.5° - \sin 0.5°) + (\sin 2.5° - \sin 1.5°) + \cdots + (\sin 90.5° - \sin 89.5°)$

$= \sin 90.5° - \sin 0.5°$

$= 2\cos 45.5° \sin 45°$

$\cos 1° + \cos 2° + \cos 3° + \ldots + \cos 90° = \dfrac{\cos 45.5° \sin 45°}{\sin\frac{1°}{2}} = 56.794$

Formulae for the Trigonometric functions

Created by Mr. Francis Hung on 23 June 2008 Last updated: March 19, 2023

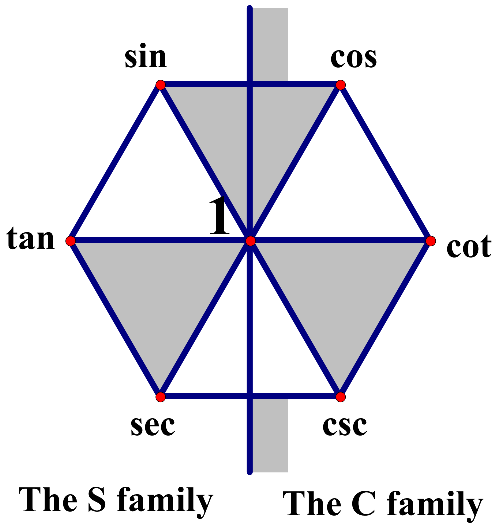
I The magic hexagon:

Along each diagonal,

$\csc\theta = \dfrac{1}{\sin\theta}$ $\quad\sin\theta = \dfrac{1}{\csc\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$ $\quad\cos\theta = \dfrac{1}{\sec\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$ $\quad\tan\theta = \dfrac{1}{\cot\theta}$



In each shaded triangle,

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

In any three adjacent vertices,

$\sin\theta = \cos\theta\cdot\tan\theta$ $\quad\cos\theta = \sin\theta\cdot\cot\theta$

$\cot\theta = \cos\theta\cdot\csc\theta$ $\quad\csc\theta = \cot\theta\cdot\sec\theta$

$\sec\theta = \tan\theta\cdot\csc\theta$ $\quad\tan\theta = \sin\theta\cdot\sec\theta$

II General Solutions

$\sin\theta = \sin\alpha$, $\theta = 180°n + (-1)^n\alpha$ $\quad\theta = n\pi + (-1)^n\alpha$, where n is an integer.

$\cos\theta = \cos\alpha$, $\theta = 360°n \pm \alpha$ $\quad\theta = 2n\pi \pm \alpha$, where n is an integer.

$\tan\theta = \tan\alpha$, $\theta = 180°n + \alpha$ $\quad\theta = n\pi + \alpha$, where n is an integer.

III Compound Angle Formulae

$\sin(A + B) = \sin A\cos B + \cos A\sin B$ SC + CS

$\sin(A - B) = \sin A\cos B - \cos A\sin B$ SC – CS

$\cos(A + B) = \cos A\cos B - \sin A\sin B$ CC – SS

$\cos(A - B) = \cos A\cos B + \sin A\sin B$ CC + SS

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$ $\quad\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan A\tan C}$$

IV Multiple angles

$\sin 2\theta = 2\sin\theta\cdot\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

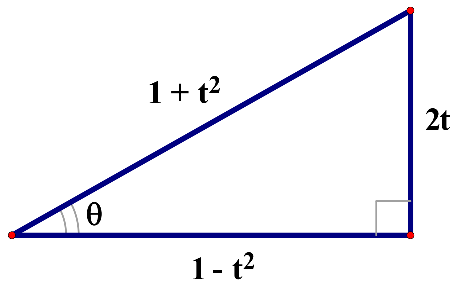
$\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ $\quad\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

V Half angles

Let $t=\tan\frac{\theta}{2}$, then $\sin\theta=\frac{2t}{1+t^2}$

$$\cos\theta=\frac{1-t^2}{1+t^2}$$

$$\tan\theta=\frac{2t}{1-t^2}$$



$$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}=\frac{\sin x}{1+\cos x}=\frac{1-\cos x}{\sin x}$$

VI Sum and Product

Sum

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Product

$$\sin X\cos Y=\frac{1}{2}\left[\sin(X+Y)+\sin(X-Y)\right]$$

$$\cos X\sin Y=\frac{1}{2}\left[\sin(X+Y)-\sin(X-Y)\right]$$

$$\cos X\cos Y=\frac{1}{2}\left[\cos(X+Y)+\cos(X-Y)\right]$$

$$\sin X\sin Y=-\frac{1}{2}\left[\cos(X+Y)-\cos(X-Y)\right]$$

VII Differentiation: In each shaded triangle's edge,

$$\text{DS}=+\ \frac{d\sin x}{dx}=\cos x \qquad \text{DC}=-\ \frac{d\cos x}{dx}=-\sin x$$

$$\frac{d\tan x}{dx}=\sec^2 x \qquad \frac{d\cot x}{dx}=-\csc^2 x$$

$$\frac{d\sec x}{dx}=\sec x\tan x \qquad \frac{d\csc x}{dx}=-\csc x\cot x$$

Integration: the inverse process of differentiation

$$\int\cos x\,dx=\sin x+C \qquad \int\sin x\,dx=-\cos x+C$$

$$\int\sec^2 x\,dx=\tan x+C \qquad \int\csc^2 x\,dx=-\cot x+C$$

$$\int\sec x\tan x\,dx=\sec x+C \qquad \int\csc x\cot x\,dx=-\csc x+C$$

Supplementary Exercise on Trigonometry

Created by Mr. Francis Hung Last updated: March 19, 2023

1. Prove that $\cos^2(A + \theta) + \cos^2(B + \theta) + 2\cos(A - B)\sin(A + \theta)\sin(B + \theta)$ is independent of θ.
2. Prove that, if $\sin(\alpha + \beta) = k\sin(\alpha - \beta)$, then $(k + 1)\cot\alpha = (k - 1)\cot\beta$.
3. If $x\sin\theta + y\cos\theta = \sin\varphi$ and $x\cos\theta - y\sin\theta = \cos\varphi$, express x in terms of θ and φ as simple as possible.
4. Without using tables or calculators, find the values of $\tan 15°$ and $\tan 22\frac{1}{2}°$.
5. If $\cos\theta + \cos\varphi = x$ and $\sin\theta + \sin\varphi = y$, prove that $\cos\frac{1}{2}(\theta - \varphi) = \pm\frac{1}{2}\sqrt{x^2 + y^2}$.
6. Prove the identities: $\sin^2 A + \sin^2 B - \sin^2(A - B) = 2\sin A\sin B\cos(A - B)$, and
$$\frac{\tan 3A - 2\tan 2A + \tan A}{4(\tan 3A - \tan 2A)} = \sin^2 A.$$
7. In any triangle ABC, prove that
   (*a*) $b^2\sin(C - A) = (c^2 - a^2)\sin B$,
   (*b*) $a^2 - (b - c)^2\cos^2\frac{A}{2} = (b + c)^2\sin^2\frac{A}{2}$,
   (*c*) $\tan(\frac{A}{2} + B) = \frac{c + b}{c - b}\tan\frac{A}{2}$.
8. Solve completely the triangle ABC in which $a = 2.718$, $b = 3.142$, $A = 54°18'$. Show that there are two possible triangles and find their areas.
9. If A, B, C are the angles of a triangle, prove that
$$(\sin B - \cos B)^2 + (\sin C - \cos C)^2 - (\sin A - \cos A)^2 = 1 - 4\sin A\cos B\cos C.$$
10. Given that $(1 + \cos A)(1 + \cos B)(1 + \cos C)(1 + \cos D) = p\sin A\sin B\sin C\sin D$, prove that $(1 - \cos A)(1 - \cos B)(1 - \cos C)(1 - \cos D) = \frac{1}{p}\sin A\sin B\sin C\sin D$.
11. If A, B, C are the angles of a triangle, using sine rule to prove $\begin{cases} a = b\cos C + c\cos B \\ b = c\cos A + a\cos C \\ c = a\cos B + b\cos A \end{cases} \cdots (*)$

    Hence, solve the system (*) and express $\cos A$, $\cos B$, $\cos C$ in terms of a, b and c.
12. If $\theta + \varphi = \frac{1}{4}\pi$, prove that $(1 + \tan\theta)(1 + \tan\varphi) = 2$. Deduce the value of $\tan\frac{1}{8}\pi$.
13. Establish the identity $\sin\theta(\cos 2\theta + \cos 4\theta + \cos 6\theta) = \sin 3\theta\cos 4\theta$.
    Prove that, if $x = \cos 3\theta + \sin 3\theta$ and $y = \cos\theta - \sin\theta$, then $x - y = 2y\sin 2\theta$.
14. By expressing $(3 + \cos\theta)\operatorname{cosec}\theta$ in terms of $\tan\frac{1}{2}\theta$ $(= t)$, show that this expression cannot have any value between $-2\sqrt{2}$ and $2\sqrt{2}$.
15. By projection of the sides of an equilateral triangle onto a certain line, or otherwise, prove that
$$\cos\theta + \cos(\theta + \tfrac{2}{3}\pi) + \cos(\theta + \tfrac{4}{3}\pi) = 0,$$
    and find the value of $\sin\theta + \sin(\theta + \frac{2}{3}\pi) + \sin(\theta + \frac{4}{3}\pi)$.
16. If $\tan\alpha = k\tan\beta$, show that $(k - 1)\sin(\alpha + \beta) = (k + 1)\sin(\alpha - \beta)$.
    Show that, if the equation $\tan x = k\tan(x - \alpha)$ has real solutions (in x), $(k - 1)^2$ is not less than $(k + 1)^2\sin^2\alpha$.
    Solve the equation when $k = -2$ and $\alpha = 30°$, give your answer in general solution.
17. (a) If $\sin\alpha + \cos\alpha = 2a$, form the quadratic equation whose roots are $\sin\alpha$ and $\cos\alpha$.
    (b) Solve the equation and find the general solution of x:
$$\cos^2 x + \cos x - \sin x - \sin^2 x = 0.$$

18. Let $y = \sin\theta\,(3\sin\theta - \sin 2\alpha) + \cos\theta\,(3\cos\theta - \cos 2\alpha)$ $(0° < \alpha < 90°)$.
   (a) Find the general solution of θ such that y has a minimum value and find this value.
   (b) Find the general solution of θ such that y has a maximum value and find this value.
   (c) Find also the maximum and minimum value of the expression
   $$\sin\alpha\,(3\sin\alpha - \sin 2\theta) + \cos\alpha\,(3\cos\alpha - \cos 2\theta) \quad (0° < \alpha < 90°).$$
19. If α, β are two distinct roots of the equation $a\cos\theta + b\sin\theta = c$, prove that
   $$\frac{a}{b}\sin(\alpha+\beta) - \cos(\alpha+\beta) = 1.$$
20. Prove the identity $\dfrac{\cos(2\theta+\varphi)+\cos(2\varphi+\theta)}{2\cos(\theta+\varphi)-1} = \dfrac{\cos(2\theta-\varphi)+\cos(2\varphi-\theta)}{2\cos(\theta-\varphi)-1}$.
21. Prove the identities
   (a) $\sin^2(2\theta+\varphi) + \sin^2(2\varphi+\theta) - \sin^2(\theta-\varphi) = 2\cos(\theta-\varphi)\sin(2\theta+\varphi)\sin(2\varphi+\theta)$,
   (b) $\tan(3A - \frac{3}{4}\pi)\tan(A + \frac{1}{4}\pi) = \dfrac{1+2\sin 2A}{1-2\sin 2A}$,
   (c) $(\cos A + \cos B + \cos C)^2 + (\sin A + \sin B + \sin C)^2 = 1 + 8\cos\frac{1}{2}(B-C)\cos\frac{1}{2}(C-A)\cos\frac{1}{2}(A-B)$,
   (d) $\dfrac{\tan 3A}{\tan A} = \dfrac{2\cos 2A+1}{2\cos 2A-1}$,
   (e) $\sin^2(A+\theta) + \sin^2(B+\theta) = 1 - \cos(A-B)\cos(A+B+2\theta)$.

End of Exercise

1. $\cos^2(A+\theta)+\cos^2(B+\theta)+2\cos(A-B)\sin(A+\theta)\sin(B+\theta)$

$= \frac{1}{2}[1+\cos 2(A+\theta)]+\frac{1}{2}[1+2\cos 2(B+\theta)]-\cos(A-B)[\cos(A+B+2\theta)-\cos(A-B)]$

$= 1+\frac{1}{2}[\cos 2(A+\theta)+\cos 2(B+\theta)]-\cos(A-B)\cos(A+B+2\theta)+\cos^2(A-B)$

$= 1+\cos(A+B+2\theta)\cos(A-B)-\cos(A-B)\cos(A+B+2\theta)+\cos^2(A-B)$

$= 1+\cos^2(A-B)$

After simplification, the expression does not carry θ, which is independent of θ.

2. $\sin(\alpha+\beta)=k\sin(\alpha-\beta)$, then $(k+1)\cot\alpha=(k-1)\cot\beta$.

$\sin\alpha\cos\beta+\cos\alpha\sin\beta=k\sin\alpha\cos\beta-k\cos\alpha\sin\beta$

$k\cos\alpha\sin\beta+\cos\alpha\sin\beta=k\sin\alpha\cos\beta-\sin\alpha\cos\beta$

$(k+1)\cos\alpha\sin\beta=(k-1)\sin\alpha\cos\beta$

$(k+1)\cot\alpha=(k-1)\cot\beta$

3. $x\sin\theta+y\cos\theta=\sin\varphi$ ....(1)

$x\cos\theta-y\sin\theta=\cos\varphi$ ....(2)

(1) $\times\sin\theta$: $x\sin^2\theta+y\sin\theta\cos\theta=\sin\theta\sin\varphi$

+) (2) $\times\cos\theta$: $x\cos^2\theta-y\sin\theta\cos\theta=\cos\theta\cos\varphi$

$x=\sin\theta\sin\varphi+\cos\theta\cos\varphi$

$=\cos(\varphi-\theta)$

4. Using the formula $\tan x=\dfrac{\sin 2x}{1+\cos 2x}$ (proof: $\dfrac{\sin 2x}{1+\cos 2x}=\dfrac{2\sin x\cos x}{1+2\cos^2 x-1}=\tan x$)

$$\tan 15^\circ=\frac{\sin 30^\circ}{1+\cos 30^\circ}=\frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}=\frac{1}{2+\sqrt{3}}=\frac{1}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}=2-\sqrt{3}$$

$$\tan 22\frac{1}{2}^\circ=\frac{\sin 45^\circ}{1+\cos 45^\circ}=\frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}=\frac{1}{\sqrt{2}+1}=\frac{1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1}=\sqrt{2}-1$$

Alternatively, using the formula $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$

$\tan 30^\circ=\dfrac{2\tan 15^\circ}{1-\tan^2 15^\circ}$

$\dfrac{1}{\sqrt{3}}=\dfrac{2\tan 15^\circ}{1-\tan^2 15^\circ}$

$1-\tan^2 15^\circ=2\sqrt{3}\tan 15^\circ$

$\tan^2 15^\circ+2\sqrt{3}\tan 15^\circ-1=0$

$\tan 15^\circ=-\sqrt{3}\pm\sqrt{\left(\sqrt{3}\right)^2+1}$

$=-\sqrt{3}+2$ or $-\sqrt{3}-2$

$\because 0^\circ<15^\circ<90^\circ$ $\therefore \tan 15^\circ>0\Rightarrow -\sqrt{3}-2$ is rejected

$\tan 15^\circ=2-\sqrt{3}$

$\tan 22\frac{1}{2}^\circ$ can be found in a similar way.

5. $\cos\theta + \cos\varphi = x$ ....(1)

$\sin\theta + \sin\varphi = y$ ....(2)

$$\begin{array}{ll} (1)^2 & \cos^2\theta + \cos^2\varphi + 2\cos\theta\cos\varphi = x^2 \\ +)\quad (2)^2 & \sin^2\theta + \sin^2\varphi + 2\sin\theta\sin\varphi = y^2 \\ \hline & 1 + 1 + 2\cos(\theta-\varphi) = x^2 + y^2 \end{array}$$

$2[1 + \cos(\theta - \varphi)] = x^2 + y^2$

$2[1 + 2\cos^2\frac{1}{2}(\theta-\varphi) - 1] = x^2 + y^2$

$4\cos^2\frac{1}{2}(\theta-\varphi) = x^2 + y^2$

$\cos\frac{1}{2}(\theta-\varphi) = \pm\frac{1}{2}\sqrt{x^2+y^2}$

6. RHS $= 2\sin A\sin B\cos(A-B)$

$= -[\cos(A+B) - \cos(A-B)]\cos(A-B)$

$= -\cos(A+B)\cos(A-B) + \cos^2(A-B)$

$= -\frac{1}{2}(\cos 2A + \cos 2B) + 1 - \sin^2(A-B)$

$= -\frac{1}{2}(1 - 2\sin^2 A + 1 - 2\sin^2 B) + 1 - \sin^2(A-B)$

$= \sin^2 A + \sin^2 B - \sin^2(A-B) =$ LHS

You may try to prove the identity from the left side.

LHS $= \dfrac{\tan 3A - 2\tan 2A + \tan A}{4(\tan 3A - \tan 2A)}$

$= \dfrac{1}{4} - \dfrac{\tan 2A - \tan A}{4(\tan 3A - \tan 2A)}$

$= \dfrac{1}{4} - \dfrac{\frac{\sin(2A-A)}{\cos 2A\cos A}}{\frac{4\sin(3A-2A)}{\cos 3A\cos 2A}}$ (note that $\tan\alpha - \tan\beta = \dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta} = \dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}$

$= \dfrac{1}{4} - \dfrac{\sin A\cos 3A}{4\sin A\cos A}$ $\qquad = \dfrac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$)

$= \dfrac{1}{4} - \dfrac{4\cos^3 A - 3\cos A}{4\cos A}$

$= \frac{1}{4}[1 - (4\cos^2 A - 3)]$

$= \frac{1}{4}(4 - 4\cos^2 A)$

$= \sin^2 A =$ RHS

7. In ΔABC, $A + B + C = 180°$, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k \Rightarrow a = k\sin A$, $b = k\sin B$, $c = k\sin C$.

(*a*) $\dfrac{b^2}{c^2 - a^2} = \dfrac{(k\sin B)^2}{(k\sin C)^2 - (k\sin A)^2}$

$= \dfrac{\sin^2 B}{\sin^2 C - \sin^2 A}$

$= \dfrac{\sin^2 B}{(\sin C - \sin A)(\sin C + \sin A)}$

$$=\frac{\sin^2 B}{2\cos\frac{C+A}{2}\sin\frac{C-A}{2}\cdot 2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}$$

$$=\frac{\sin^2 B}{2\sin\frac{A+C}{2}\cos\frac{A+C}{2}\cdot 2\sin\frac{C-A}{2}\cos\frac{C-A}{2}}$$

$$=\frac{\sin^2 B}{\sin(A+C)\sin(C-A)}$$

$$=\frac{\sin^2 B}{\sin(180^\circ-B)\sin(C-A)}$$

$$=\frac{\sin^2 B}{\sin B\sin(C-A)}$$

$=\dfrac{\sin B}{\sin(C-A)}$. Hence result follows.

(*b*) $a^2-(b-c)^2\cos^2\frac{1}{2}A$

$$=a^2-(b-c)^2\cdot\frac{1+\cos A}{2}$$

$=\frac{1}{2}\left[2b^2+2c^2-4bc\cos A-\left(b^2-2bc+c^2\right)(1+\cos A)\right]$, by cosine rule

$$=\frac{1}{2}\left[2b^2+2c^2-4bc\cos A-\left(b^2-2bc+c^2+b^2\cos A-2bc\cos A+c^2\cos A\right)\right]$$

$$=\frac{1}{2}\left[b^2+2bc+c^2-\left(b^2\cos A+2bc\cos A+c^2\cos A\right)\right]$$

$$=\frac{1}{2}\left[(b+c)^2-(b+c)^2\cos A\right]$$

$$=(b+c)^2\frac{1-\cos A}{2}$$

$$=(b+c)^2\sin^2\frac{1}{2}A$$

(*c*) $\dfrac{c+b}{c-b}\tan\dfrac{A}{2}$

$$=\frac{k\sin C+k\sin B}{k\sin C-k\sin B}\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$=\frac{2\sin\frac{B+C}{2}\cos\frac{C-B}{2}}{2\cos\frac{B+C}{2}\sin\frac{C-B}{2}}\cdot\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$=\dfrac{2\cos\frac{A}{2}\cos\frac{C-B}{2}}{2\sin\frac{A}{2}\sin\frac{C-B}{2}}\cdot\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$, note that $\sin\dfrac{B+C}{2}=\sin\dfrac{180^\circ-A}{2}=\cos\dfrac{A}{2}$,

$$=\frac{1}{\tan\frac{C-B}{2}}$$ $\cos\frac{B+C}{2}$ can be simplified similarly.

$$=\frac{1}{\tan\frac{180^\circ-A-B-B}{2}}$$

$$=\frac{1}{\tan\left(90^\circ-\frac{A}{2}-B\right)}$$

$= \tan(\frac{1}{2}A + B)$, this is known as tangent rule.

8. $a = 2.718$, $b = 3.142$, $A = 54°18'$: SSA

By cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$

$2.718^2 = 3.142^2 + c^2 - 2\times3.142\, c \cos 54°18'$

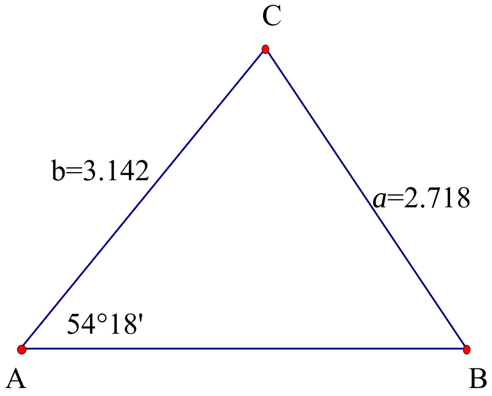
$c^2 - 3.666973\, c + 2.48464 = 0$

$c = 2.77$ or 0.90 ($B = 69.8°$ or $110.2°$, $C = 55.9°$ or $15.6°$)

So, there are two possible triangles.

$c = 2.77$, area $=\frac{1}{2} bc \sin A = \frac{1}{2}\times3.142\times2.77 \sin 54.3° = 3.53$

$c = 0.90$, area $=\frac{1}{2}\times3.142\times0.90 \sin 54.3° = 1.14$



9. $(\sin B - \cos B)^2 + (\sin C - \cos C)^2 - (\sin A - \cos A)^2$

$= \sin^2 B - 2 \sin B \cos B + \cos^2 B + \sin^2 C - 2 \sin C \cos C + \cos^2 B - \sin^2 A + 2 \sin A \cos A - \cos^2 A$

$= 1 - \sin 2B + 1 - 2 \sin C \cos C - (1 - \sin 2A)$

$= 1 + \sin 2A - \sin 2B - 2 \sin C \cos C$

$= 1 + 2 \cos(A + B) \sin(A - B) - 2 \sin C \cos C$

$= 1 - 2 \cos C \sin(A - B) - 2 \sin C \cos C$

$= 1 - 2 \cos C[\sin(A - B) + \sin C]$

$= 1 - 4 \cos C \sin\frac{A-B+C}{2}\cos\frac{A-B-C}{2}$

$= 1 - 4 \cos C \sin(90° - B) \cos(A - 90°)$

$= 1 - 4 \sin A \cos B \cos C$.

10. Given that $(1 + \cos A)(1 + \cos B)(1 + \cos C)(1 + \cos D) = p \sin A \sin B \sin C \sin D$

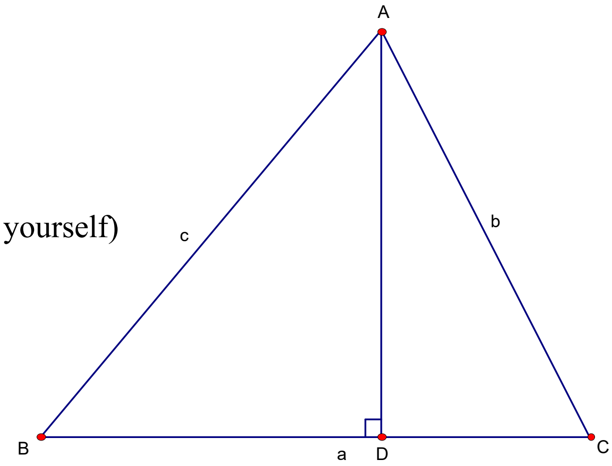
$(1 - \cos A)(1 - \cos B)(1 - \cos C)(1 - \cos D)$

$$=\frac{(1-\cos A)(1-\cos B)(1-\cos C)(1-\cos D)(1+\cos A)(1+\cos B)(1+\cos C)(1+\cos D)}{(1+\cos A)(1+\cos B)(1+\cos C)(1+\cos D)}$$

$$=\frac{(1-\cos^2 A)(1-\cos^2 B)(1-\cos^2 C)(1-\cos^2 D)}{p\sin A\sin B\sin C\sin D}$$

$$=\frac{\sin^2 A\sin^2 B\sin^2 C\sin^2 D}{p\sin A\sin B\sin C\sin D}$$

$$=\frac{1}{p}\sin A\sin B\sin C\sin D.$$



11. In the triangle on the right,
$a = BC = BD + DC = c \cos B + b \cos C \cdots\cdots (1)$

$b = AC = AE + EC = c \cos A + a \cos C \cdots\cdots (2)$

Similarly $c = a \cos B + b \cos A \cdots\cdots (3)$ (do this part yourself)

These equations are known as projection formulae.

In (2) $\cos C = \dfrac{b - c\cos A}{a} \cdots\cdots (4)$

In (3) $\cos B = \dfrac{c - b\cos A}{a} \cdots\cdots (5)$

Put (4) and (5) into (1)

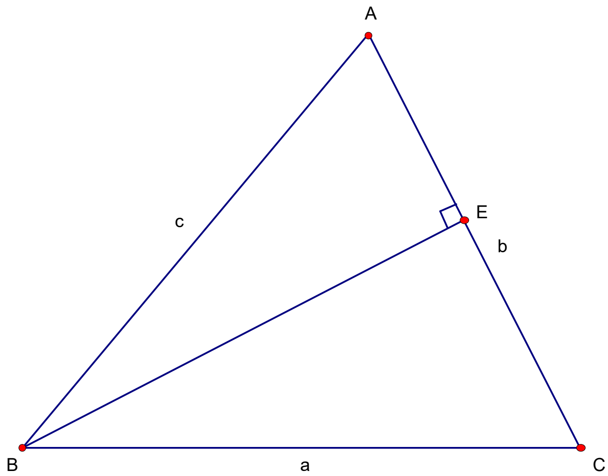
$a = c\cdot\dfrac{c - b\cos A}{a} + b\cdot\dfrac{b - c\cos A}{a}$

$\Rightarrow a^2 = c^2 + b^2 - 2bc \cos A$

Similarly $b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

(Do the substitution yourself.)



12. $\theta + \varphi = \frac{1}{4}\pi$, $\tan(\theta + \varphi) = \tan \frac{1}{4}\pi = 1$

$\dfrac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi} = 1$

$\tan \theta + \tan \varphi = 1 - \tan \theta \tan \varphi$

$1 + \tan \theta + \tan \varphi + \tan \theta \tan \varphi = 2$

$(1 + \tan \theta)(1 + \tan \varphi) = 2$

Let $\theta = \varphi = \frac{1}{8}\pi$, then $\theta + \varphi = \frac{1}{4}\pi$

By the above result, $(1 + \tan \theta)(1 + \tan \varphi) = 2$

$\Rightarrow (1 + \tan \theta)^2 = 2$

$1 + \tan \theta = \pm\sqrt{2}$

$\tan \frac{1}{8}\pi = \sqrt{2} - 1$ (reject $-\sqrt{2} - 1$)

13. $\sin \theta(\cos 2\theta + \cos 4\theta + \cos 6\theta)$

$= \sin \theta(\cos 4\theta + 2\cos 4\theta \cos 2\theta)$

$= \cos 4\theta \sin \theta(2 \cos 2\theta + 1)$

$= \cos 4\theta\ [2 (1 - 2\sin^2 \theta) + 1] \sin \theta$

$= \cos 4\theta(3 \sin \theta - 4 \sin^3 \theta)$

$= \sin 3\theta \cos 4\theta$

If $x = \cos 3\theta + \sin 3\theta$ and $y = \cos \theta - \sin \theta$,

$x - y = \cos 3\theta + \sin 3\theta - \cos \theta + \sin \theta$

$= 4 \cos^3 \theta - 3 \cos \theta + 3 \sin \theta - 4 \sin^3 \theta - \cos \theta + \sin \theta$

$= 4 \cos^3 \theta - 4 \cos \theta + 4 \sin \theta - 4 \sin^3 \theta$

$= 4 \cos \theta(\cos^2 \theta - 1) + 4 \sin^2 \theta(1 - \sin^2 \theta)$

$= -4 \cos \theta \sin^2 \theta + 4 \sin \theta \cos^2 \theta$

$= 4 \sin \theta \cos \theta(\cos \theta - \sin \theta)$

$= 2y \sin 2\theta$.

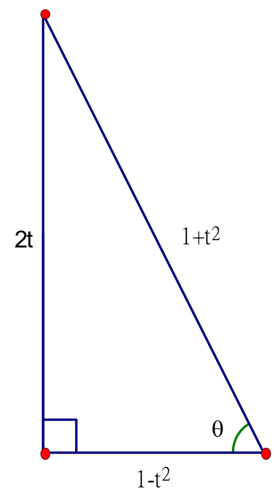
14. Let $E = (3 + \cos\theta)\ \text{cosec}\ \theta$

$= (3 + \frac{1-t^2}{1+t^2})\frac{1+t^2}{2t}$

$= \frac{3+3t^2+1-t^2}{2t}$

$= \frac{4+2t^2}{2t}$

$= \frac{2+t^2}{t}$



$Et = 2 + t^2$

$t^2 - Et + 2 = 0$

$\because t = \tan\frac{\theta}{2}$ can be any real number, $\Delta \ge 0$

$E^2 - 4\times 2 \ge 0$

$(E + 2\sqrt{2})(E - 2\sqrt{2}) \ge 0$

$\Rightarrow E \le -2\sqrt{2}$ or $E \ge 2\sqrt{2}$

This expression cannot have any value between $-2\sqrt{2}$ and $2\sqrt{2}$.

15. ΔPQR is an equilateral with QR inclined to the horizontal at an angle θ.

Suppose $PQ = QR = RP = 1$ unit.

The projection of QR on horizontal is MT, the projection of PQ on horizontal is MN, the projection of PR on horizontal is NT.

$MT = QR \cos\theta = \cos\theta$

$MN = PQ \cos(\frac{1}{3}\pi + \theta) = \cos(\frac{1}{3}\pi + \theta)$

$NT = PR \cos(\frac{1}{3}\pi - \theta)$

$\because MT = MN + NT$

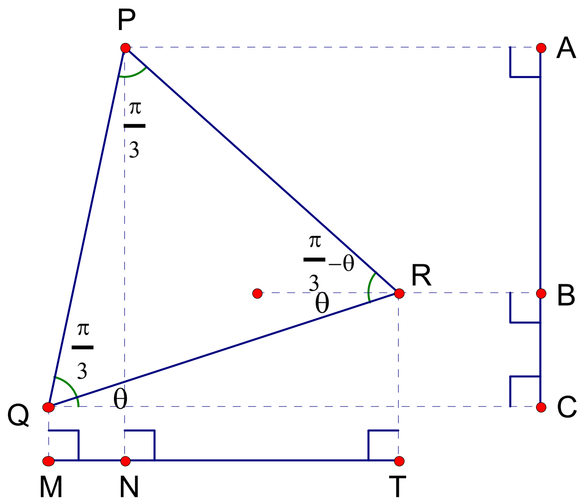
$\cos\theta = \cos(\frac{1}{3}\pi + \theta) + \cos(\frac{1}{3}\pi - \theta)$

$\cos\theta - \cos(\frac{1}{3}\pi + \theta) - \cos(\frac{1}{3}\pi - \theta) = 0$

$\cos\theta + \cos(\frac{1}{3}\pi + \theta + \pi) + \cos(\frac{1}{3}\pi - \theta - \pi) = 0$

$\cos\theta + \cos(\frac{4}{3}\pi + \theta) + \cos(-\frac{2}{3}\pi - \theta) = 0$

$\cos\theta + \cos(\theta + \frac{2}{3}\pi) + \cos(\theta + \frac{4}{3}\pi) = 0$



Using the projection of ΔPQR onto the vertical,

$AB = PR \sin(\frac{1}{3}\pi - \theta) = \sin(\frac{1}{3}\pi - \theta)$

$BC = QR \sin\theta = \sin\theta$

$AC = PQ \sin(\frac{1}{3}\pi + \theta) = \sin(\frac{1}{3}\pi + \theta)$

$AB + BC = AC$

$\sin(\frac{1}{3}\pi - \theta) + \sin\theta = \sin(\frac{1}{3}\pi + \theta)$

$\sin(\frac{1}{3}\pi - \theta) + \sin\theta - \sin(\frac{1}{3}\pi + \theta) = 0$

$\sin\theta + \sin[\pi - (\frac{1}{3}\pi - \theta)] + \sin(\frac{1}{3}\pi + \theta + \pi) = 0$

$\sin\theta + \sin(\theta + \frac{2}{3}\pi) + \sin(\theta + \frac{4}{3}\pi) = 0$

16. $\tan\alpha = k\tan\beta$

$\frac{\sin\alpha}{\cos\alpha} = \frac{k\sin\beta}{\cos\beta}$

$\sin\alpha\cos\beta = k\sin\beta\cos\alpha$

$\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] = \frac{k}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

$\sin(\alpha - \beta) + k\sin(\alpha - \beta) = k\sin(\alpha + \beta) - \sin(\alpha + \beta)$

$(k - 1)\sin(\alpha + \beta) = (k + 1)\sin(\alpha - \beta)$

Given $\tan x = k \tan (x - \alpha)$

$(k - 1) \sin (2x - \alpha) = (k + 1) \sin \alpha$

$\sin(2x-\alpha)=\frac{(k+1)}{(k-1)}\sin\alpha$

It has real solutions (in x) $\Rightarrow -1 \le \frac{(k+1)}{(k-1)}\sin\alpha \le 1$

$\frac{(k+1)^2}{(k-1)^2}\sin^2\alpha \le 1$

$(k + 1)^2 \sin^2 \alpha \le (k - 1)^2$

$(k - 1)^2$ is not less than $(k + 1)^2 \sin^2 \alpha$.

When $k = -2$ and $\alpha = 30°$,

$(-2 - 1) \sin (2x - 30°) = (-2 + 1) \sin 30°$

$\sin (2x - 30°) = \frac{1}{6}$

$2x - 30° = 180°n + (-1)^n\, 9.594°$

$2x = 180°n + (-1)^n\, 9.594° + 30°$

$x = 90°n + (-1)^n\, 4.797° + 15°$, $n = 0, \pm1, \pm2, \pm3,$ ....

17. (a) If $\sin \alpha + \cos \alpha = 2a$

Squaring: $\sin^2 \alpha + \cos^2 \alpha + 2 \sin \alpha \cos \alpha = 4a^2$

$2 \sin \alpha \cos \alpha = 4a^2 - 1$

$\sin \alpha \cos \alpha = 2a^2 - \frac{1}{2}$

sum of roots $= 2a$, product of roots $= 2a^2 - \frac{1}{2}$

Quadratic equation whose roots are $\sin \alpha$, $\cos \alpha$ is $x^2 - 2ax + 2a^2 - \frac{1}{2} = 0$

$2x^2 - 4ax + 4a^2 - 1 = 0$

(b) $\cos^2 x + \cos x - \sin x - \sin^2 x = 0$

$\cos^2 x - \sin^2 x + \cos x - \sin x = 0$

$(\cos x + \sin x)(\cos x - \sin x) + \cos x - \sin x = 0$

$(\cos x - \sin x)(\cos x + \sin x + 1) = 0$

$\cos x - \sin x = 0$ or $\cos x + \sin x = -1$

$\tan x = 1$ or $\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x=-\frac{1}{\sqrt{2}}$

$x = 180°n + 45°$ or $\cos (x - 45°) = \cos 135°$

$x = 180°n + 45°$ or $x - 45° = 360°n \pm 135°$

$x = 180°n + 45°$ or $x = 360°n + 180°$ or $360°n - 90°$, $n = 0, \pm1, \pm2, \pm3, \cdots$.

18. Let $y = \sin \theta (3 \sin \theta - \sin 2\alpha) + \cos \theta (3 \cos \theta - \cos 2\alpha)$ $(0° < \alpha < 90°)$.

$=3 \sin^2 \theta - \sin \theta \sin 2\alpha + 3 \cos^2 \theta - \cos \theta \cos 2\alpha$

$= 3 - (\sin \theta \sin 2\alpha + \cos \theta \cos 2\alpha)$

$= 3 - \cos(\theta - 2\alpha)$

(a) When y has a minimum value, $\cos(\theta - 2\alpha) = 1$

$\theta - 2\alpha = 360°n$

$\theta = 360°n + 2\alpha$, $n = 0, \pm1, \pm2, \pm3, \cdots$.

(b) When y has a maximum value, $\cos(\theta - 2\alpha) = -1$

$\theta - 2\alpha = 360°n + 180°$

$\theta = 360°n + 180° + 2\alpha$, $n = 0, \pm1, \pm2, \pm3, \cdots$.

(c) Interchange the role of α and θ, we have

$\sin\alpha\,(3\sin\alpha - \sin 2\theta) + \cos\alpha\,(3\cos\alpha - \cos 2\theta) = 3 - \cos(\alpha - 2\theta)$

Maximum value is $3 - (-1) = 4$

Minimum value is $3 - 1 = 2$

19. $a\cos\theta + b\sin\theta = c$, α, β are two distinct roots.

$a\cos\alpha + b\sin\alpha = c \cdots\cdots (1)$

$a\cos\beta + b\sin\beta = c \cdots\cdots (2)$

(1) = (2) $a\cos\alpha + b\sin\alpha = a\cos\beta + b\sin\beta$

$a(\cos\alpha - \cos\beta) = b(\sin\beta - \sin\alpha)$

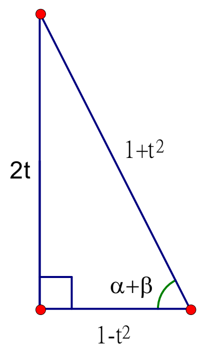
$$-2a\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = -2b\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$\because$ α, β are distinct $\therefore \sin\frac{\alpha-\beta}{2} \neq 0$

$$a\sin\frac{\alpha+\beta}{2} = b\cos\frac{\alpha+\beta}{2}$$

$$\tan\frac{\alpha+\beta}{2} = \frac{b}{a}$$

Using the formula for half angle: $t = \tan\frac{\alpha+\beta}{2}$



$$\frac{a}{b}\sin(\alpha+\beta) - \cos(\alpha+\beta) = \frac{a}{b}\cdot\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}$$

$$= \frac{a}{b}\cdot\left[\frac{2\left(\frac{b}{a}\right)}{1+\left(\frac{b}{a}\right)^2}\right] - \frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2}$$

$$= \frac{a}{b}\cdot\frac{2ab}{a^2+b^2} - \frac{a^2-b^2}{a^2+b^2}$$

$$= \frac{2a^2 - a^2 + b^2}{a^2+b^2} = 1$$

20. $\dfrac{\cos(2\theta+\varphi)+\cos(2\varphi+\theta)}{2\cos(\theta+\varphi)-1} = \dfrac{\cos(2\theta-\varphi)+\cos(2\varphi-\theta)}{2\cos(\theta-\varphi)-1}$ is equivalent to

$[2\cos(\theta-\varphi) - 1][\cos(2\theta+\varphi) + \cos(2\varphi+\theta)] = [2\cos(\theta+\varphi) - 1][\cos(2\theta-\varphi) + \cos(2\varphi-\theta)]$

LHS $= 2\cos(2\theta+\varphi)\cos(\theta-\varphi) + 2\cos(2\varphi+\theta)\cos(\theta-\varphi) - [\cos(2\theta+\varphi) + \cos(2\varphi+\theta)]$

$= \cos 3\theta + \cos(\theta+2\varphi) + \cos(2\theta+\varphi) + \cos 3\varphi - \cos(2\theta+\varphi) - \cos(2\varphi+\theta)$

$= \cos 3\theta + \cos 3\varphi$

RHS $= 2\cos(2\theta-\varphi)\cos(\theta+\varphi) + 2\cos(2\varphi-\theta)\cos(\theta+\varphi) - [\cos(2\theta-\varphi) + \cos(2\varphi-\theta)]$

$= \cos 3\theta + \cos(\theta-2\varphi) + \cos 3\varphi + \cos(\varphi-2\theta) - \cos(2\theta-\varphi) - \cos(2\varphi-\theta)$

$= \cos 3\theta + \cos 3\varphi$

$\therefore$ LHS = RHS

21. (a) $2\cos(\theta-\varphi)\sin(2\theta+\varphi)\sin(2\varphi+\theta)$

$= -\cos(\theta-\varphi)[\cos(3\theta+3\varphi) - \cos(\theta-\varphi)]$

$= -\cos(\theta-\varphi)\cos(3\theta+3\varphi) + \cos^2(\theta-\varphi)$

$= -\frac{1}{2}[\cos(4\theta+2\varphi) + \cos(2\theta+4\varphi)] + 1 - \sin^2(\theta-\varphi)$

$= -\frac{1}{2}[1 - 2\sin^2(2\theta+\varphi) + 1 - 2\sin^2(\theta+2\varphi)] + 1 - \sin^2(\theta-\varphi)$

$= \sin^2(2\theta+\varphi) + \sin^2(2\varphi+\theta) - \sin^2(\theta-\varphi)$ (similar to Q6(*a*))

(b) $\tan(3A - \frac{3}{4}\pi)\tan(A + \frac{1}{4}\pi)$

$$= \frac{\sin\left(3A - \frac{3\pi}{4}\right)\sin\left(A + \frac{\pi}{4}\right)}{\cos\left(3A - \frac{3\pi}{4}\right)\cos\left(A + \frac{\pi}{4}\right)}$$

$$= \frac{-\frac{1}{2}\left[\cos\left(4A - \frac{\pi}{2}\right) - \cos(2A - \pi)\right]}{\frac{1}{2}\left[\cos\left(4A - \frac{\pi}{2}\right) + \cos(2A - \pi)\right]}$$

$$= \frac{-(\sin 4A + \cos 2A)}{\sin 4A - \cos 2A}$$

$$= -\frac{2\sin 2A\cos 2A + \cos 2A}{2\sin 2A\cos 2A - \cos 2A}$$

$$= -\frac{2\sin 2A + 1}{2\sin 2A - 1}$$

$$= \frac{1 + 2\sin 2A}{1 - 2\sin 2A}$$

(c) $(\cos A + \cos B + \cos C)^2 + (\sin A + \sin B + \sin C)^2$

$= \cos^2 A + \cos^2 B + \cos^2 C + \sin^2 A + \sin^2 B + \sin^2 C + 2(\cos A\cos B + \sin A\sin B$
$+ \cos B\cos C + \sin B\sin C + \cos C\cos A + \sin A\sin C)$

$= 3 + 2[\cos(A - B) + \cos(B - C) + \cos(C - A)]$

$$= 3 + 4\cos\frac{A - C}{2}\cos\frac{A - 2B + C}{2} + 4\cos^2\frac{C - A}{2} - 2$$

$$= 1 + 4\cos\frac{C - A}{2}\left[\cos\frac{A - 2B + C}{2} + \cos\frac{C - A}{2}\right]$$

$$= 1 + 8\cos\frac{C - A}{2}\cos\frac{C - B}{2}\cos\frac{A - B}{2} = \text{RHS}$$

(d) $\dfrac{\tan 3A}{\tan A}$

$$= \frac{\sin 3A}{\cos 3A}\cdot\frac{\cos A}{\sin A}$$

$$= \frac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A}\cdot\frac{\cos A}{\sin A}$$

$$= \frac{3 - 4\sin^2 A}{4\cos^2 A - 3}$$

$$= \frac{3 - 2(1 - \cos 2A)}{2(1 + \cos 2A) - 3}$$

$$= \frac{2\cos 2A + 1}{2\cos 2A - 1},$$

(e) $\sin^2(A + \theta) + \sin^2(B + \theta)$

$$= \frac{1}{2}[1 - \cos(2A + 2\theta)] + \frac{1}{2}[1 - \cos(2B + 2\theta)]$$

$$= 1 - \frac{1}{2}[\cos(2A + 2\theta) + \cos(2B + 2\theta)]$$

$$= 1 - \cos(A - B)\cos(A + B + 2\theta)$$