Cosine formula

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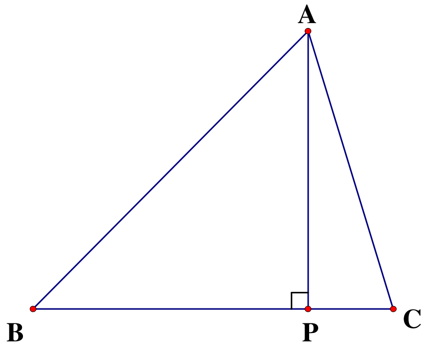
In ΔABC, let P be the foot of perpendicular from A onto BC.

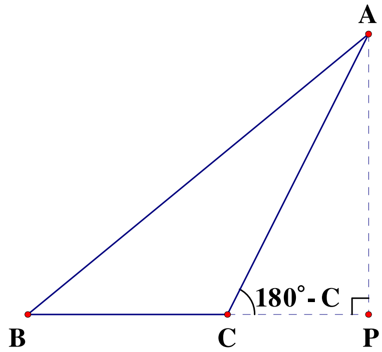
Case 1 $\angle C < 90°$



$BC = BP + PC$

$a = c \cos B + b \cos C$ ……(1)

Case 2 $\angle C > 90°$



$BC = BP - PC$

$a = c \cos B - b \cos(180° - C)$

$a = c \cos B + b \cos C$ ……(1)

When $\angle C = 90°$, the formula $a = c \cos B + b \cos C$ is true obviously.

Similarly, $b = c \cos A + a \cos C$ ……(2)

$c = a \cos B + b \cos A$ ……(3)

From (1) $\cos B = \dfrac{a - b\cos C}{c}$ ……(4)

From (2) $\cos A = \dfrac{b - a\cos C}{c}$ ……(5)

Substitute (4) and (5) into (3),

$$c = a\cdot\frac{a - b\cos C}{c} + b\cdot\frac{b - a\cos C}{c}$$

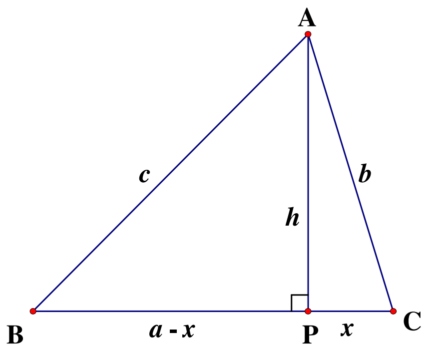
$c^2 = a^2 + b^2 - 2ab \cos C$.

Since a, b and c are symmetric variables, we can derive the similar formulae:

$b^2 = a^2 + c^2 - 2ac \cos B$ and $a^2 = b^2 + c^2 - 2bc \cos A$.

Second proof: In ΔABC, let P be the foot of perpendicular from A onto BC. Let $CP = x$, $AP = h$

Case 1 $\angle C < 90°$



$BP = a - x$, $x = b \cos C$

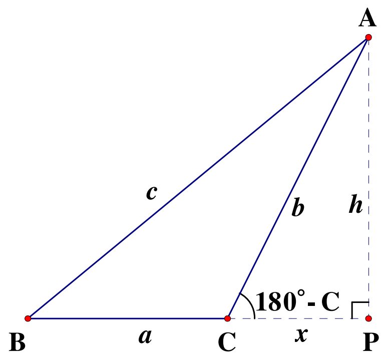
$h^2 = b^2 - x^2 = c^2 - (a - x)^2$

$b^2 - x^2 = c^2 - a^2 + 2ax - x^2$

$c^2 = a^2 + b^2 - 2ax$

$c^2 = a^2 + b^2 - 2ab \cos C$

Case 2 $\angle C > 90°$



$BP = a + x$, $x = b \cos(180° - C) = -b \cos C$

$h^2 = b^2 - x^2 = c^2 - (a + x)^2$

$b^2 - x^2 = c^2 - a^2 - 2ax - x^2$

$c^2 = a^2 + b^2 + 2ax$

$c^2 = a^2 + b^2 - 2ab \cos C$

When $\angle C = 90°$, the formula $c^2 = a^2 + b^2 - 2ab \cos C$ is true obviously.

We can also change cosine as the subject of the formula:

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$;

$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$ and

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Third proof: vector dot product method.

$c^2 = \overrightarrow{AB} \cdot \overrightarrow{AB} = (\overrightarrow{OB} - \overrightarrow{OA}) \cdot (\overrightarrow{OB} - \overrightarrow{OA})$

$= \left|\overrightarrow{OB}\right|^2 + \left|\overrightarrow{OA}\right|^2 - 2\,\overrightarrow{OB} \cdot \overrightarrow{OA}$

$= a^2 + b^2 - 2ab \cos C$

Example 1 (SAS) Given $a = 6$, $b = 5$, $C = 60°$, find c.

$c^2 = 6^2 + 5^2 - 2\times6\times5\times\cos 60°$

$c = \sqrt{31}$

Example 2 (SSS) Given that $a = 6$, $b = 5$, $c = 7$, find C.

$\cos C = \dfrac{6^2 + 5^2 - 7^2}{2\times6\times5} = \dfrac{1}{5}$

$C = 78.5°$

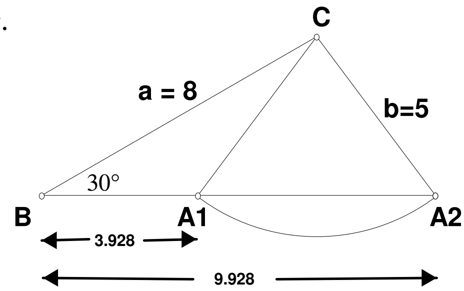
Example 3 (SSA) Given that $a = 8$, $b = 5$, $B = 30°$,find c.

$5^2 = 8^2 + c^2 - 2\times8\times c \cos 30°$

$c^2 - 8\sqrt{3}\,c + 39 = 0$, a quadratic equation in c.

$c = 4\sqrt{3} \pm 3$

Please see the right figure for reference.



Example 4

In ΔABC, if $\dfrac{b+c}{11} = \dfrac{c+a}{12} = \dfrac{a+b}{13}$, prove that $\dfrac{\sin A}{7} = \dfrac{\sin B}{6} = \dfrac{\sin C}{5}$ and $\dfrac{\cos A}{7} = \dfrac{\cos B}{19} = \dfrac{\cos C}{25}$.

$b + c = 11k \cdots\cdots (1)$, $c + a = 12k \cdots\cdots (2)$, $a + b = 13k \cdots\cdots (3)$

(2) + (3) – (1): $2a = 14k \Rightarrow a = 7k$

Sub. $a = 7k$ into (3): $7k + b = 13k \Rightarrow b = 6k$

Sub. $a = 7k$ into (2): $c + 7k = 12k \Rightarrow c = 5k$

By sine rule, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} \Rightarrow \dfrac{\sin A}{7k} = \dfrac{\sin B}{6k} = \dfrac{\sin C}{5k} \Rightarrow \dfrac{\sin A}{7} = \dfrac{\sin B}{6} = \dfrac{\sin C}{5}$

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{(6k)^2 + (5k)^2 - (7k)^2}{2(6k)(5k)} = \dfrac{1}{5}$

$\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{(7k)^2 + (5k)^2 - (6k)^2}{2(7k)(5k)} = \dfrac{19}{35}$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab} = \dfrac{(7k)^2 + (6k)^2 - (5k)^2}{2(7k)(6k)} = \dfrac{5}{7}$

$\cos A : \cos B : \cos C = \dfrac{1}{5} : \dfrac{19}{35} : \dfrac{5}{7} = 7 : 19 : 25$

$\dfrac{\cos A}{7} = \dfrac{\cos B}{19} = \dfrac{\cos C}{25}$

Example 5 In ΔABC, D is the mid-point of BC. Prove that $\sin \angle ADB = \dfrac{2b \sin C}{\sqrt{2b^2+2c^2-a^2}}$.

Produce AD with its own length to E so that $AD = DE$.
$ABEC$ is a parallelogram (diagonal bisect each other)

Let $\angle ADB = \theta$, $BD = DC = \dfrac{a}{2}$

$BE = b$, $CE = c$ (opp. sides of //-gram)

Apply sine rule on ΔADC

$$\frac{AD}{\sin C} = \frac{b}{\sin\theta}$$

$$\therefore \sin\theta = \frac{b\sin C}{AC}$$

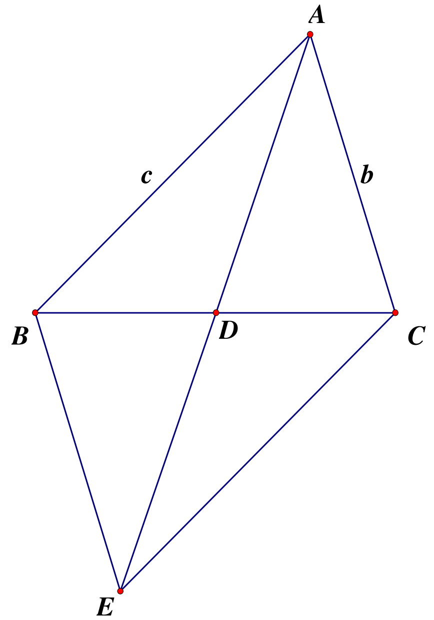
$$= \frac{2b\sin C}{AE}$$

$$= \frac{2b\sin C}{\sqrt{b^2+c^2-2bc\cos\angle ACE}} \quad \text{(cosine rule on } \Delta ACE)$$

$$= \frac{2b\sin C}{\sqrt{b^2+c^2-2bc\cos(180^\circ - A)}} \quad (\because ABEC \text{ is a //-gram})$$

$$= \frac{2b\sin C}{\sqrt{b^2+c^2+(b^2+c^2-a^2)}} \quad \text{(cosine rule on } \Delta ABC)$$

$$= \frac{2b\sin C}{\sqrt{2b^2+2c^2-a^2}}$$



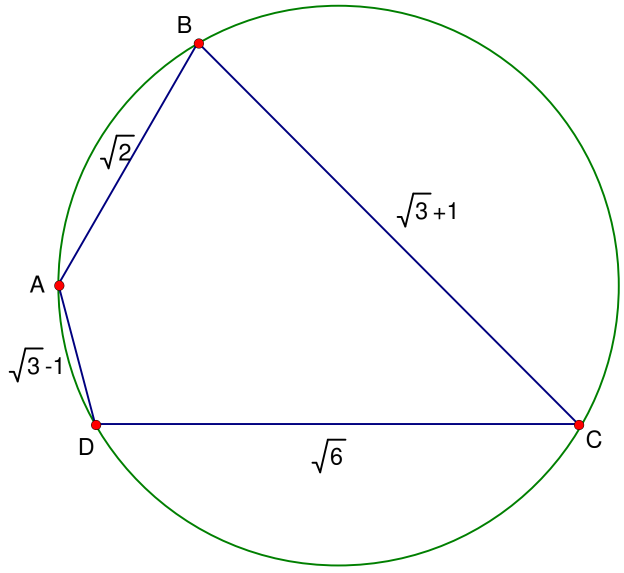
Classwork 1 In ΔABC, if $\angle A : \angle B : \angle C = 1 : 2 : 3$, find $a : b : c$. [$1 : \sqrt{3} : 2$]

Classwork 2 If $(b + c) : (c + a) : (a + b) = 5 : 6 : 7$, find $\sin A : \sin B : \sin C$ and $\cos A : \cos B : \cos C$. [$4 : 3 : 2$, $-4 : 11 : 14$]

Classwork 3 In ΔABC, if $\angle A = 36°$, $b = 2$, $a = \sqrt{5} - 1$, find c. [2 or $\sqrt{5}-1$ (=1.236)]

Classwork 4 Let a, b, c be the 3 sides of ΔABC such that $a^2 - a - 2b - 2c = 0$ and $a + 2b - 2c + 3 = 0$. Find the greatest angle of the triangle. [120°]

Classwork 5 In the figure, $AB = \sqrt{2}$, $BC = \sqrt{3} + 1$, $CD = \sqrt{6}$, $AD = \sqrt{3} - 1$, find $\angle A$, $\angle D$.



[$\angle A = 135°$, $\angle D = 105°$]

Solution to classwork 1

$\angle A : \angle B : \angle C = 1 : 2 : 3$

$\angle A = k$, $\angle B = 2k$, $\angle C = 3k$

$\angle A + \angle B + \angle C = k + 2k + 3k = 180°$ ($\angle$ sum of Δ)

$k = 30°$

$\angle A = 30°$, $\angle B = 60°$, $\angle C = 90°$

By sine formula, $a : b : c = \sin A : \sin B : \sin C$

$= \sin 30° : \sin 60° : \sin 90°$

$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

$= 1 : \sqrt{3} : 2$

Solution to classwork 2

$(b + c) : (c + a) : (a + b) = 5 : 6 : 7$

$$\frac{b+c}{5} = \frac{c+a}{6} = \frac{a+b}{7} = k$$

$b + c = 5k \cdots (1)$

$c + a = 6k \cdots (2)$

$a + b = 7k \cdots (3)$

(2) + (3) – (1): $2a = 8k \Rightarrow a = 4k$

Sub. $a = 4k$ into (3): $4k + b = 7k \Rightarrow b = 3k$

Sub. $a = 4k$ into (2): $c + 4k = 6k \Rightarrow c = 2k$

$\sin A : \sin B : \sin C = a : b : c = 4 : 3 : 2$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(3k)^2 + (2k)^2 - (4k)^2}{2(3k)(2k)} = \frac{-1}{4}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{(4k)^2 + (2k)^2 - (3k)^2}{2(4k)(2k)} = \frac{11}{16}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(4k)^2 + (3k)^2 - (2k)^2}{2(4k)(3k)} = \frac{7}{8}$$

$$\cos A : \cos B : \cos C = \frac{-1}{4} : \frac{11}{16} : \frac{7}{8} = \frac{-4}{16} : \frac{11}{16} : \frac{14}{16} = -4 : 11 : 14$$

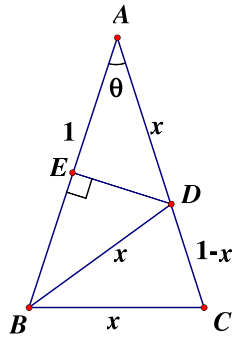
Solution to classwork 3

Find cos 36° without using triple angle formula.

Consider the following triangle ΔABC.

Given $AB = AC = 1$. D is a point lying on AC such that $AD = BD = BC = x$.

Let $\angle A = \theta$, $CD = 1 - x$.



Then $\angle ABD = \theta$, (base ∠s isos. Δ)

$\angle BDC = 2\theta$ (ext. ∠ of Δ)

$\angle ACB = 2\theta$ (base ∠s isos. Δ)

$\angle ABC = 2\theta$ (base ∠s isos. Δ)

$\angle CBD = 2\theta - \theta = \theta$

$\Delta ABC \sim \Delta BCD$ (equiangular)

In ΔABC, $\theta + 2\theta + 2\theta = 180°$ (∠ sum of Δ)

$\theta = 36°$

$\frac{AB}{BC} = \frac{BC}{CD}$ (corr. sides, ~Δ's)

$\frac{1}{x} = \frac{x}{1-x}$

$1 + x = x^2$

$x^2 - x - 1 = 0$

$x = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$ (< 0, rejected)

Draw $DE \perp AB$ as shown. Then $\Delta ADE \cong \Delta BDE$ (R.H.S.)

$AE = ED = \frac{1}{2}$ (corr. sides, ≅ Δs)

$$\cos 36° = \frac{AE}{AD} = \frac{\frac{1}{2}}{x} = \frac{\frac{1}{2}}{\frac{1+\sqrt{5}}{2}} = \frac{1}{1+\sqrt{5}} = \frac{1}{1+\sqrt{5}} \cdot \frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{1+\sqrt{5}}{4}$$

In ΔABC, if $\angle A = 36°$, $b = 2$, $a = \sqrt{5} - 1$, find c.

$a^2 = b^2 + c^2 - 2bc \cos 36°$

$$\left(\sqrt{5}-1\right)^2 = 2^2 + c^2 - 2(2)c \cdot \frac{\sqrt{5}+1}{4}$$

$$5 - 2\sqrt{5} + 1 = 4 + c^2 - \left(\sqrt{5}+1\right)c$$

$$c^2 - \left(\sqrt{5}+1\right)c + 2\left(\sqrt{5}-1\right) = 0$$

$$(c-2)\left[c - \left(\sqrt{5}-1\right)\right] = 0$$

$c = 2$ or $\sqrt{5} - 1$ (=1.236)

Solution to classwork 4

$$\begin{cases} b+c=\frac{1}{2}\left(a^2-a\right)\cdots\cdots(1) \\ b-c=-\frac{1}{2}(3+a)\cdots\cdots(2) \end{cases}$$

$$\frac{(1)+(2)}{2}: b=\frac{1}{4}\left(a^2-a-3-a\right)=\frac{1}{4}\left(a^2-2a-3\right)$$

$$\frac{(1)-(2)}{2}: c=\frac{1}{4}\left(a^2-a+3+a\right)=\frac{1}{4}\left(a^2+3\right)$$

$$\cos\angle C=\frac{a^2+b^2-c^2}{2ab}=\frac{a^2+\frac{1}{16}\left(a^2-2a-3\right)^2-\frac{1}{16}\left(a^2+3\right)^2}{2a\times\frac{1}{4}\left(a^2-2a-3\right)}$$

$$=\frac{a^2+\frac{1}{16}\left(a^2-2a-3+a^2+3\right)\left(a^2-2a-3-a^2-3\right)}{\frac{1}{2}a\left(a^2-2a-3\right)}$$

$$=\frac{a^2+\frac{1}{16}\left(2a^2-2a\right)(-2a-6)}{\frac{1}{2}a\left(a^2-2a-3\right)}$$

$$=\frac{a^2-\frac{1}{4}\left(a^2-a\right)(a+3)}{\frac{1}{2}a\left(a^2-2a-3\right)}$$

$$=\frac{4a^2-\left(a^3-a^2+3a^2-3a\right)}{2a\left(a^2-2a-3\right)}$$

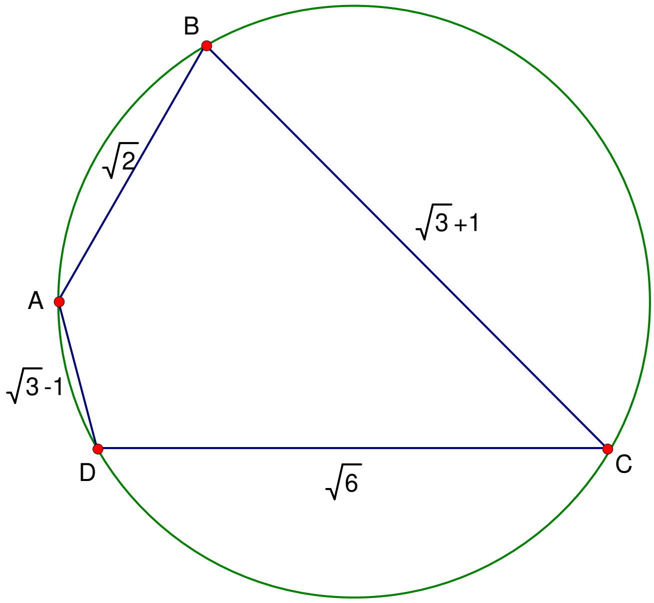
$$=\frac{4a^2-\left(a^3-a^2+3a^2-3a\right)}{2a\left(a^2-2a-3\right)}$$

$$=\frac{-\left(a^3-2a^2-3a\right)}{2\left(a^3-2a^2-3a\right)}=-\frac{1}{2}$$

The largest angle = $\angle C$ = 120°

Solution to Classwork 5

Classwork 5 In the figure, $AB=\sqrt{2}$, $BC=\sqrt{3}+1$, $CD=\sqrt{6}$, $AD=\sqrt{3}-1$, find $\angle A$, $\angle D$.



[$\angle A$ = 135°, $\angle D$ = 105°]

$BD^2=\left(\sqrt{3}-1\right)^2+\left(\sqrt{2}\right)^2-2\left(\sqrt{3}-1\right)\left(\sqrt{2}\right)\cos A=\left(\sqrt{3}+1\right)^2+\left(\sqrt{6}\right)^2-2\left(\sqrt{3}+1\right)\left(\sqrt{6}\right)\cos C$

$\angle A + \angle C$ = 180° (opp. $\angle$s cyclic quadrilateral) $\therefore$ cos C = cos(180° – A) = –cos A

$3-2\sqrt{3}+1+2-2\left(\sqrt{6}-\sqrt{2}\right)\cos A=3+2\sqrt{3}+1+6+2\left(3\sqrt{2}+\sqrt{6}\right)\cos A$

$-4\sqrt{3}-4=4\left(\sqrt{6}+\sqrt{2}\right)\cos A$

$\cos A=-\dfrac{\sqrt{3}+1}{\sqrt{6}+\sqrt{2}}=-\dfrac{\sqrt{3}+1}{\sqrt{2}\left(\sqrt{3}+1\right)}=-\dfrac{1}{\sqrt{2}}$

$\angle A$ = 135°

$AC^2=\left(\sqrt{3}-1\right)^2+\left(\sqrt{6}\right)^2-2\left(\sqrt{3}-1\right)\left(\sqrt{6}\right)\cos D=\left(\sqrt{3}+1\right)^2+\left(\sqrt{2}\right)^2-2\left(\sqrt{3}+1\right)\left(\sqrt{2}\right)\cos B$

$\angle B + \angle D$ = 180° (opp. $\angle$s cyclic quadrilateral) $\therefore$ cos B = cos(180° – D) = –cos D

$3-2\sqrt{3}+1+6-2\left(3\sqrt{2}-\sqrt{6}\right)\cos D=3+2\sqrt{3}+1+2+2\left(\sqrt{6}+\sqrt{2}\right)\cos D$

$-4\sqrt{3}+4=8\sqrt{2}\cos D$

$\cos D=-\dfrac{\sqrt{3}-1}{2\sqrt{2}}=-\dfrac{\sqrt{6}-\sqrt{2}}{4}$

$\angle D$ = 105°