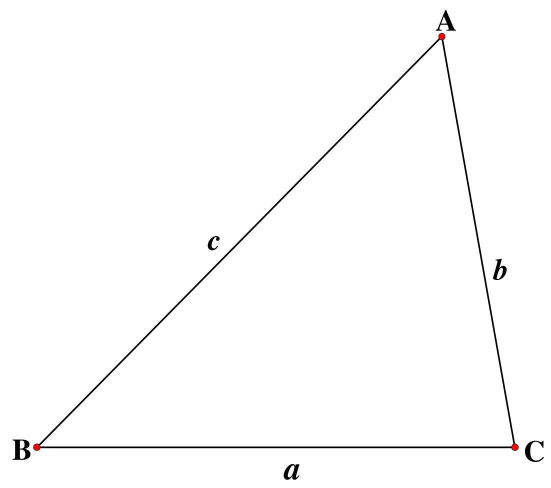
Heron's formula --- the area of a triangle, given 3 sides

Created by Mr. Francis Hung Last updated: July 26, 2020

In ΔABC, let $s = \frac{1}{2}(a+b+c)$, half of a perimeter,

then the area $= \sqrt{s(s-a)(s-b)(s-c)}$.



Proof: By cosine rule $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$$1-\cos^2 C = 1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2$$

$$\begin{aligned}
\sin^2 C &= \frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2b^2} \\
&= \frac{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)}{4a^2b^2} \\
&= \frac{[(a+b)^2-c^2][c^2-(a-b)^2]}{4a^2b^2} \\
&= \frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4a^2b^2} \\
&= \frac{(a+b+c)(a+b+c-2c)(a+b+c-2b)(a+b+c-2a)}{4a^2b^2} \\
&= \frac{2s(2s-2c)(2s-2b)(2s-2a)}{4a^2b^2} = \frac{4s(s-a)(s-b)(s-c)}{a^2b^2}
\end{aligned}$$

$$\begin{aligned}
\text{area} &= \frac{1}{2}ab\sin C \\
&= \frac{1}{2}ab\sqrt{\frac{4s(s-a)(s-b)(s-c)}{a^2b^2}} = \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

Example Let $a = 5$, $b = 6$, $c = 7$. then $s = \frac{1}{2}(5+6+7) = 9$

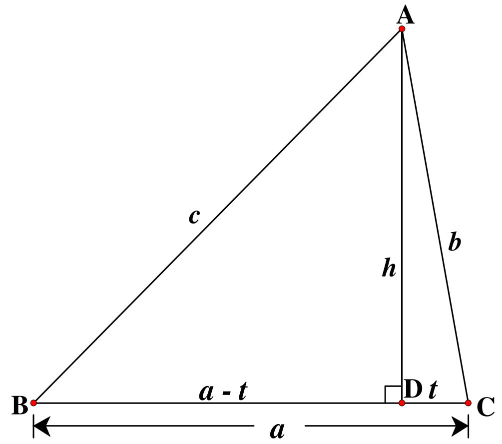
$s-a = 9-5 = 4$, $s-b = 9-6 = 3$, $s-c = 9-7 = 2$

area $= \sqrt{9\times4\times3\times2} = 6\sqrt{6}$

In ΔABC, let $s=\frac{1}{2}(a+b+c)$, half of a perimeter, then the area $=\sqrt{s(s-a)(s-b)(s-c)}$.

Proof: (**method 2**)

Case 1 $\angle C < 90°$ and $\angle B < 90°$



Let D be the foot of perpendicular from A to BC.

Let $CD = t$, $BD = a - t$, let $AD = h$.

$h^2 = b^2 - t^2 = c^2 - (a - t)^2$ (Pythagoras' theorem)

$b^2 - t^2 = c^2 - (a^2 - 2at + t^2)$

$b^2 = c^2 - a^2 + 2at$

$t=\frac{a^2+b^2-c^2}{2a}$

$h^2 = b^2 - t^2 = (b + t)(b - t)$

$=\left(b+\frac{a^2+b^2-c^2}{2a}\right)\left(b-\frac{a^2+b^2-c^2}{2a}\right)$

$=\left(\frac{a^2+2ab+b^2-c^2}{2a}\right)\left[\frac{c^2-(a^2-2ab+b^2)}{2a}\right]$

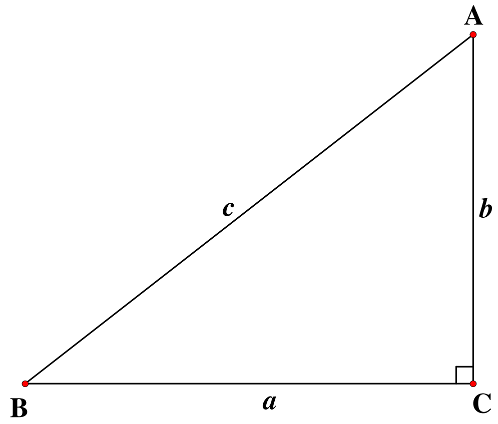
$=\frac{1}{(2a)^2}\left[(a+b)^2-c^2\right]\left[c^2-(a-b)^2\right]$

$=\frac{1}{(2a)^2}(a+b+c)(a+b-c)(c+a-b)(c-a+b)=\frac{1}{(2a)^2}(2s)(2s-2c)(2s-2b)(2s-2a)$

$=\frac{4}{a^2}s(s-a)(s-b)(s-c)\Rightarrow h=\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$

Area of $\Delta ABC=\frac{1}{2}ah=\frac{1}{2}a\times\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{s(s-a)(s-b)(s-c)}$

Case 2 $\angle C = 90°$ or $\angle B = 90°$ (WLOG assume $\angle C = 90°$)



Area $=\frac{1}{2}ab$

$c^2 = a^2 + b^2$ (Pythagoras' theorem)

$\sqrt{s(s-a)(s-b)(s-c)}$

$=\sqrt{\frac{a+b+c}{2}\cdot\frac{b+c-a}{2}\cdot\frac{a+c-b}{2}\cdot\frac{a+b-c}{2}}$

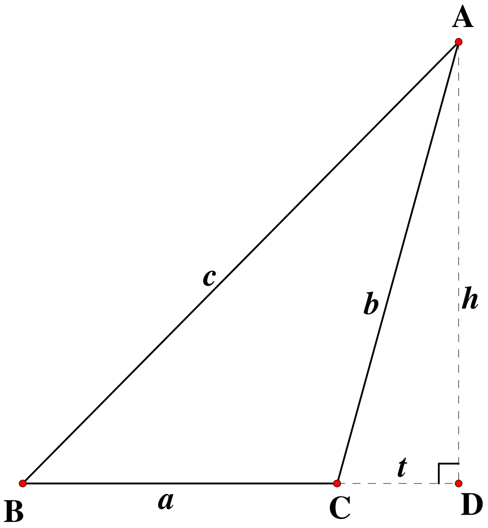
$=\frac{1}{4}\sqrt{\left[(b+c)^2-a^2\right]\left[a^2-(c-b)^2\right]}$

$=\frac{1}{4}\sqrt{\left[(b^2+2bc+c^2)-(c^2-b^2)\right]\left[(c^2-b^2)-(c^2-2bc+b^2)\right]}$

$=\frac{1}{4}\sqrt{(2b^2+2bc)(2bc-2b^2)}=\frac{1}{2}\sqrt{(bc+b^2)(bc-b^2)}=\frac{1}{2}\sqrt{b^2c^2-b^4}=\frac{1}{2}\sqrt{b^2(c^2-b^2)}$

$=\frac{1}{2}ab$

Case 3 $\angle C > 90°$ or $\angle B > 90°$ (WLOG assume $\angle C > 90°$)
Let D be the foot of perpendicular from A to BC.
Let $CD = t$, $BD = a + t$, let $AD = h$.



$h^2 = b^2 - t^2 = c^2 - (a + t)^2$ (Pythagoras' theorem)

$b^2 - t^2 = c^2 - (a^2 + 2at + t^2)$

$b^2 = c^2 - a^2 - 2at$

$$t = \frac{c^2 - a^2 - b^2}{2a}$$

$$h^2 = b^2 - t^2 = (b + t)(b - t)$$

$$= \left(b + \frac{c^2 - a^2 - b^2}{2a}\right)\left(b - \frac{c^2 - a^2 - b^2}{2a}\right)$$

$$= \left[\frac{c^2 - (a^2 - 2ab + b^2)}{2a}\right]\left(\frac{a^2 + 2ab + b^2 - c^2}{2a}\right)$$

$$= \frac{1}{(2a)^2}\left[c^2 - (a-b)^2\right]\left[(a+b)^2 - c^2\right]$$

$$= \frac{1}{(2a)^2}(c+a-b)(c-a+b)(a+b+c)(a+b-c) = \frac{1}{(2a)^2}(2s-2b)(2s-2a)(2s)(2s-2c)$$

$$= \frac{4}{a^2}s(s-a)(s-b)(s-c) \Rightarrow h = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

Area of $\Delta ABC = \frac{1}{2}ah = \frac{1}{2}a \times \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a)(s-b)(s-c)}$