

Platonic solids

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I Euler's Formula on Graph Theory

Let G be a connected plane graph with V vertices, E edges and F faces, then $V - E + F = 2$.

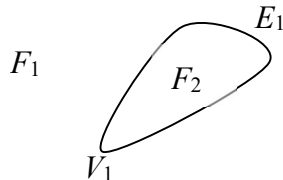
Proof: Case 1: Suppose G has no cycle. Hence G is a tree.

$$E = V - 1 \text{ (every vertices are connected.) } F = 1 \text{ (} G \text{ has no cycle.)}$$

$$V - E + F = V - (V - 1) + 1 = 2$$

Case 2: Suppose G has a cycle, use induction on the number of edges.

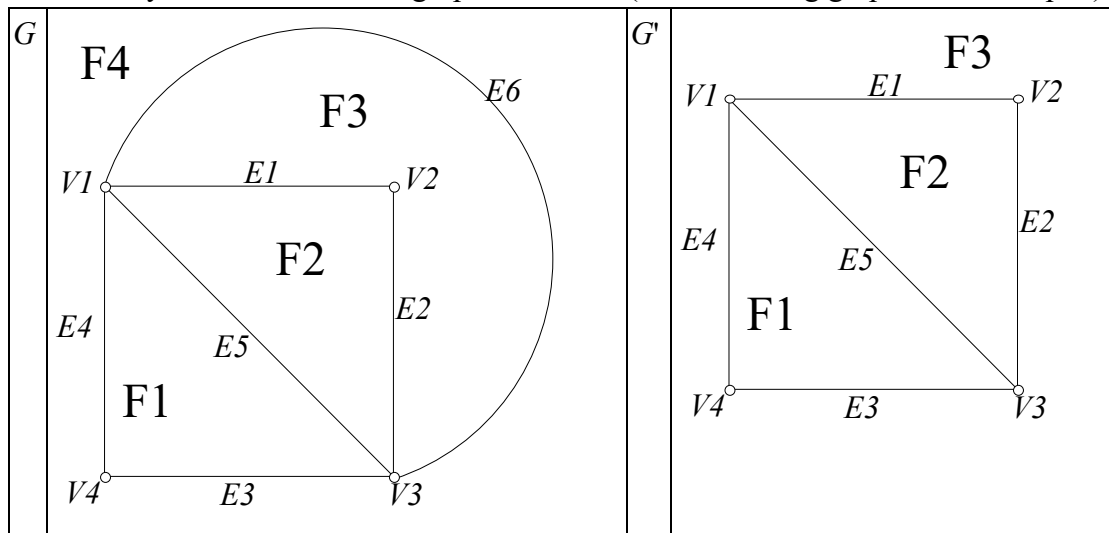
$$E = 1 \Rightarrow V = 1, F = 2 \Rightarrow V - E + F = 1 - 1 + 2 = 2.$$



It is true for $E = 1$

Suppose the formula to true for some positive integer k ($E = k$ edges).

For a connected plane graph with $k + 1$ edges (with at least one cycle). Remove an edge from this cycle to obtain a new graph called G' . (The following graph is an example.)



G' is a connected plane graph with one edge less. ($E' = k$)

$$V' = V, F' = F - 1$$

$$V' - E' + F' = 2 \text{ (by induction assumption)}$$

$$V - k + (F - 1) = 2$$

$$V - (k + 1) + F = 2$$

$$V - E + F = 2$$

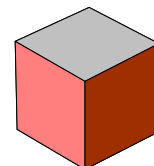
The formula is still true for $E = k + 1$.

By the principle of Mathematical Induction, Euler's formula is true for all number of edges.

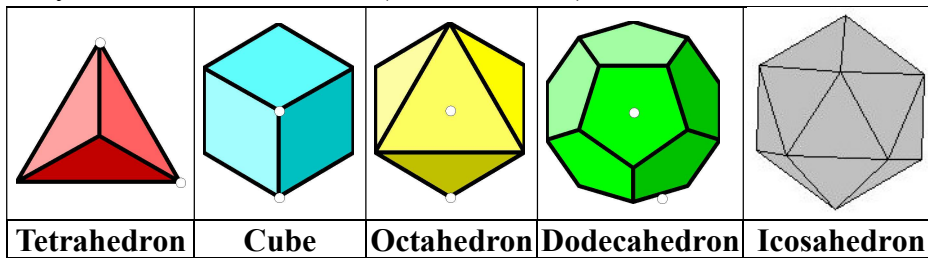
Example on Euler's formula

Given a cube, $V = 8$, $E = 12$, $F = 6$.

then $V - E + F = 2$.



Theorem There are only five Platonic solids. Each face is formed by **identical regular polygons**. They are **tetrahedron**, **cube (= hexahedron)**, **octahedron**, **icosahedron** and **dodecahedron**.



We shall prove the **existence and uniqueness of Platonic solids** by Euler's formula.

On each vertex, there are m edges ($m \geq 3$). There are V vertices, so altogether Vm edges. Each edge is connected by two vertices $\Rightarrow Vm = 2E \dots\dots (1)$

Each face is a regular polygon with n sides (edges) ($n \geq 3$). There are F faces, so altogether nF edges. Each edge is shared by two faces $\Rightarrow nF = 2E \dots\dots (2)$

From (1) and (2), substitute V and F in terms of E , m , n into Euler's Formula: $V - E + F = 2$

$$\frac{2E}{m} - E + \frac{2E}{n} = 2$$

$$E \left(\frac{2}{m} + \frac{2}{n} - 1 \right) = 2$$

$$E(2m + 2n - mn) = 2mn \dots\dots (3)$$

$\therefore E$ is a positive integer and also $\text{RHS} = 2mn$ is a positive integer

$\therefore 2m + 2n - mn$ is a positive integer.

$2m + 2n - mn = 4 - (m - 2)(n - 2)$ which is a positive integer.

$\therefore m - 2 > 0$ and $n - 2 > 0$,

Possible values of $(m - 2)(n - 2)$ may be 1, 2 or 3

When $(m - 2)(n - 2) = 1$,

$$m - 2 = 1, n - 2 = 1$$

$$m = 3, n = 3$$

Sub. into (3): $E(6 + 6 - 9) = 2(3)(3) \Rightarrow E = 6$

$$V = \frac{2E}{m} = \frac{2 \times 6}{3} = 4$$

$F = \frac{2E}{n} = 4$, the solid is a **tetrahedron**, each face is an **equilateral triangle**. ($n = 3$)

When $(m - 2)(n - 2) = 2$,

$$m - 2 = 2, n - 2 = 1 \text{ or } m - 2 = 1, n - 2 = 2$$

$$m = 4, n = 3 \text{ or } m = 3, n = 4$$

Sub. into (3): $E(8 + 6 - 12) = 2(3)(4) \Rightarrow E = 12$

when $m = 3, n = 4$

$$V = \frac{2E}{m} = \frac{2 \times 12}{3} = 8$$

$$F = \frac{2E}{n} = 6, \text{ the solid is a cube, each face is a square. } (n = 4)$$

$$\text{when } m = 4, n = 3$$

$$V = \frac{2E}{m} = \frac{2 \times 12}{4} = 6$$

$$F = \frac{2E}{n} = 8, \text{ the solid is an octahedron, each face is an equilateral triangle. } (n = 3)$$

$$\text{When } (m - 2)(n - 2) = 3,$$

$$m - 2 = 1, n - 2 = 3 \text{ or } m - 2 = 3, n - 2 = 1$$

$$m = 3, n = 5 \text{ or } m = 5, n = 3$$

$$\text{Sub. into (3): } E(6 + 10 - 15) = 2(3)(5) \Rightarrow E = 30$$

$$\text{when } m = 3, n = 5$$

$$V = \frac{2E}{m} = \frac{2 \times 30}{3} = 20$$

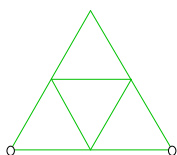
$$F = \frac{2E}{n} = 12, \text{ the solid is a dodecahedron, each face is a regular pentagon. } (n = 5)$$

$$\text{when } m = 5, n = 3$$

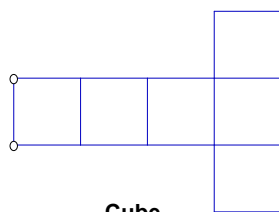
$$V = \frac{2E}{m} = \frac{2 \times 30}{5} = 12$$

$$F = \frac{2E}{n} = 20, \text{ the solid is an icosahedron, each face is an equilateral triangle. } (n = 3)$$

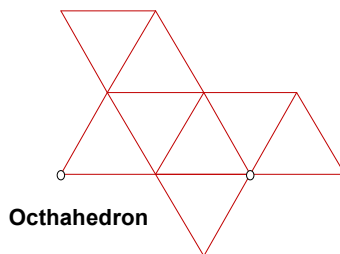
The following diagrams are **the nets (plane developments)** of the five Platonic solids.



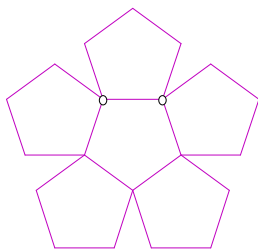
Tetrahedron



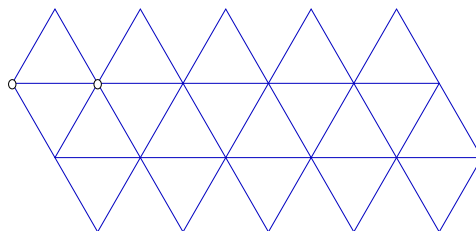
Cube



Octahedron



Dodecahedron (two needed)



Icosahedron

Let one side of the Platonic solids be a . **Express the volumes of the figures in terms of a .**

Let $ABCD$ be a regular tetrahedron with side a .

DE is the altitude from D onto the plane ABC .

E is the centroid of $\triangle ABC$.

It is easy to find the length of median of $\triangle ABC$.

$$\text{median} = a \sin 60^\circ = \frac{\sqrt{3}}{2} a$$

$$BE = \frac{2}{3} \text{median} = \frac{\sqrt{3}}{3} a$$

$$DE = \sqrt{BD^2 - BE^2} = \sqrt{a^2 - \frac{a^2}{3}} = \sqrt{\frac{2}{3}} a$$

$$\text{Volume} = \frac{1}{3} \text{base area} \times \text{height} = \frac{1}{3} \times \frac{1}{2} a^2 \sin 60^\circ \times \sqrt{\frac{2}{3}} a = \frac{\sqrt{2}}{12} a^3$$

For a square with side a , the volume is a^3

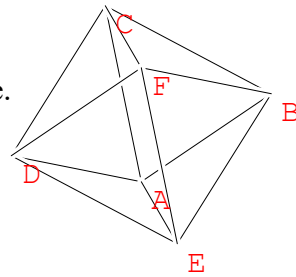
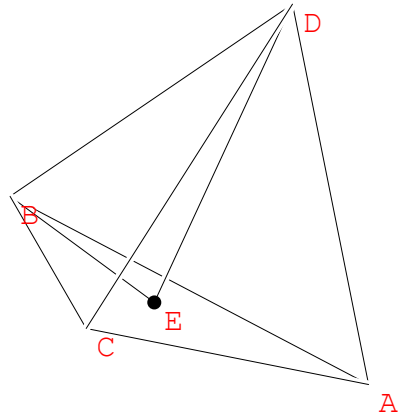
For a regular octahedron with side a ,

it can be cut into two identical right pyramids with square base.

In the figure, $BD = \sqrt{a^2 + a^2} = \sqrt{2} a$

$$\text{Height} = \sqrt{a^2 - \left(\frac{\sqrt{2}a}{2}\right)^2} = \frac{\sqrt{2}}{2} a$$

$$\text{Total volume} = 2 \times \frac{1}{3} \times a^2 \times \frac{\sqrt{2}}{2} a = \frac{\sqrt{2}}{3} a^3$$



The following figure shows a regular dodecahedron and a regular face:

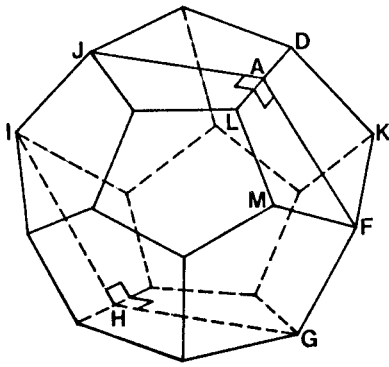


Figure 1 (a)

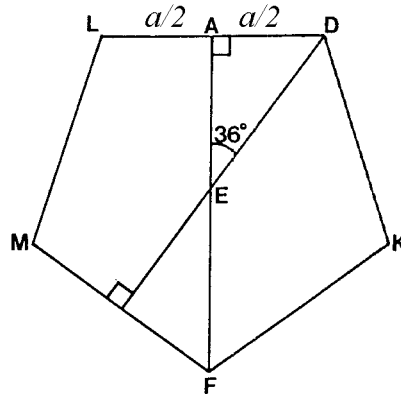


Figure 1 (b)

Let the angle between two adjacent sides be 2θ . We can use figure 1(a) and figure 1(b) to find θ .

In figure 1(b), $\triangle AED$, $AE = \frac{a}{2 \tan 36^\circ}$

$$EF = ED = \frac{a}{2 \sin 36^\circ}$$

$$AF = AE + EF = \frac{a}{2 \tan 36^\circ} + \frac{a}{2 \sin 36^\circ}$$

$$= \frac{a}{2} \left(\frac{1 + \cos 36^\circ}{\sin 36^\circ} \right) = \frac{a}{2 \tan 18^\circ}, \text{ by using the formula } t = \tan \frac{\theta}{2}.$$

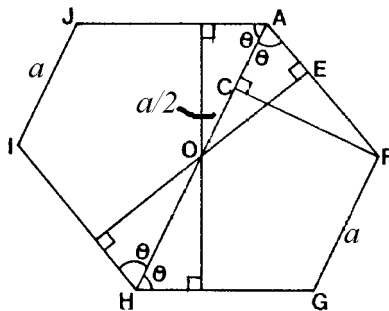


Figure 1(c)

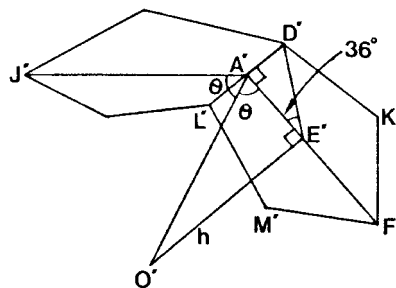


Figure 1(d)

In figure 1(c), which is the cross section of figure 1(a),

$$\cos \theta = \frac{AE}{AO} = \frac{AE}{AC + CO} = \frac{AE}{AF \cos \theta + CO} = \frac{\frac{a}{2 \tan 36^\circ}}{\frac{a \cos \theta}{2 \tan 18^\circ} + \frac{a}{2}} = \frac{\tan 18^\circ}{\tan 36^\circ \cos \theta + \tan 36^\circ \tan 18^\circ}$$

$$(\tan 36^\circ) \cos^2 \theta + (\tan 36^\circ \tan 18^\circ) \cos \theta - \tan 18^\circ = 0$$

$$\cos \theta = \frac{-\tan 18^\circ \tan 36^\circ \pm \sqrt{(\tan 18^\circ \tan 36^\circ)^2 + 4 \tan 18^\circ \tan 36^\circ}}{2 \tan 36^\circ} = -\frac{1}{2} (\tan 18^\circ \pm \tan 54^\circ)$$

$$\cos \theta > 0 \Rightarrow \cos \theta = \frac{1}{2} (\tan 54^\circ - \tan 18^\circ)$$

$$= \frac{1}{2} \left(\frac{\sin 54^\circ}{\cos 54^\circ} - \frac{\sin 18^\circ}{\cos 18^\circ} \right) = \frac{1}{2} \left(\frac{\sin 54^\circ \cos 18^\circ - \cos 54^\circ \sin 18^\circ}{\cos 54^\circ \cos 18^\circ} \right)$$

$$= \frac{1}{2} \cdot \frac{\sin(54^\circ - 18^\circ)}{\cos 54^\circ \cos 18^\circ} = \frac{1}{2 \cos 18^\circ}$$

$\theta = 58^\circ 17'$; $2\theta = 116.6^\circ$, the angle between two adjacent sides is 116.6° .

$$\begin{aligned}
 \tan \theta &= \sqrt{\sec^2 \theta - 1} = \sqrt{(2 \cos 18^\circ)^2 - 1} \\
 &= \sqrt{2(2 \cos^2 18^\circ - 1) + 1} \\
 &= \sqrt{2 \cos 36^\circ + 1} \\
 &= \sqrt{2 \times \frac{1 + \sqrt{5}}{4} + 1} \\
 &= \sqrt{\frac{6 + 2\sqrt{5}}{4}} \\
 &= \sqrt{\frac{(\sqrt{5} + 1)^2}{2^2}} \\
 &= \frac{\sqrt{5} + 1}{2}
 \end{aligned}$$

A regular dodecahedron can be regarded as 12 identical right pyramids with regular pentagonal base, side = a . If the area of the pentagonal base is B and the height of the cone is h , then the volume of the regular dodecahedron is

$$\begin{aligned}
 12\left(\frac{1}{3}\right)Bh &= 4 \times 5 \times \frac{1}{2} \times a \times AE \times OE \\
 &= 10a \times \frac{a}{2 \tan 36^\circ} \times AE \tan \theta \\
 &= 10a \times \left(\frac{a}{2 \tan 36^\circ}\right)^2 \times \tan \theta \\
 &= \frac{5a^3}{2(\sec^2 36^\circ - 1)} \times \frac{\sqrt{5} + 1}{2} \\
 &= \frac{5(\sqrt{5} + 1)a^3}{4\left[\left(\frac{4}{\sqrt{5} + 1}\right)^2 - 1\right]} \\
 &= \frac{5(\sqrt{5} + 1)^3 a^3}{4[16 - (\sqrt{5} + 1)^2]} \\
 &= \frac{5(5\sqrt{5} + 3 \times 5 + 3 \times \sqrt{5} + 1)a^3}{4[16 - (6 + 2\sqrt{5})]} \\
 &= \frac{5(8\sqrt{5} + 16)a^3}{4(10 - 2\sqrt{5})} \\
 &= \frac{5(\sqrt{5} + 2)(5 + \sqrt{5})a^3}{(5 - \sqrt{5})(5 + \sqrt{5})} \\
 &= \frac{(15 + 7\sqrt{5})a^3}{4}
 \end{aligned}$$

The following figure shows a regular icosahedron and a regular face:

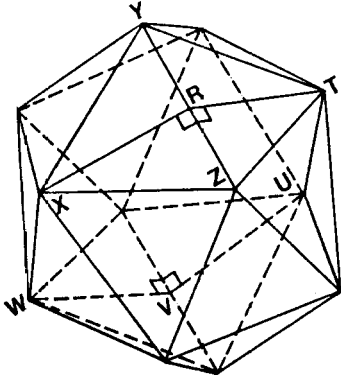


Figure 2 (a)

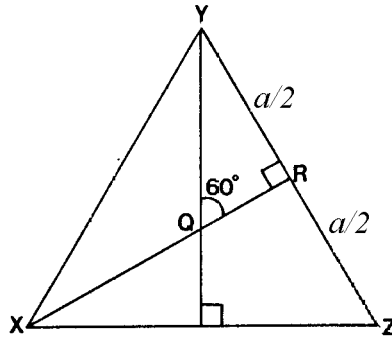


Figure 2 (b)

Let the angle between two adjacent sides be 2θ . We can use figure 2(a) and figure 2(b) to find θ .

$$\begin{aligned} \text{In figure 2(b), } \Delta YQR, RQ &= \frac{a}{2 \tan 60^\circ} = \frac{a}{2\sqrt{3}} \\ QX &= QY \\ &= \frac{a}{2 \sin 60^\circ} = \frac{a}{\sqrt{3}} \\ RX &= \frac{a}{2} \tan 60^\circ = \frac{\sqrt{3}}{2} a \end{aligned}$$

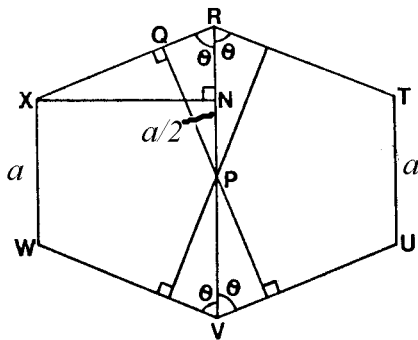


Figure 2(c)

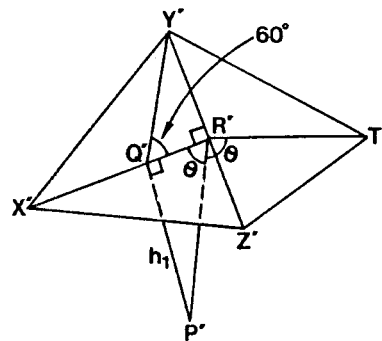


Figure 2(d)

$$\begin{aligned} \text{In the figure 2(c), } \cos \theta &= \frac{RQ}{RP} \\ &= \frac{RQ}{RN + NP} \\ &= \frac{RQ}{RX \cos \theta + NP} \\ &= \frac{\frac{a}{2\sqrt{3}}}{\frac{\sqrt{3}}{2} a \cos \theta + \frac{a}{2}} = \frac{1}{3 \cos \theta + \sqrt{3}} \end{aligned}$$

$$3 \cos^2 \theta + \sqrt{3} \cos \theta - 1 = 0$$

$$\cos \theta = \frac{-\sqrt{3} \pm \sqrt{3+12}}{6} = \frac{\sqrt{15} - \sqrt{3}}{6}$$

$$= 0.3568 \text{ or } -0.9342 \text{ (rejected)}$$

$$\theta = 69^\circ 6'$$

The angle between two adjacent sides is $2\theta = 138.189^\circ$

$$\begin{aligned}
\tan^2 \theta &= \sec^2 \theta - 1 \\
&= \frac{36}{(\sqrt{15} - \sqrt{3})^2} - 1 \\
&= \frac{36}{3(\sqrt{5} - 1)^2} - 1 \\
&= \frac{12}{6 - 2\sqrt{5}} - 1 \\
&= \frac{6}{3 - \sqrt{5}} \cdot \frac{3 + \sqrt{5}}{3 + \sqrt{5}} - 1 \\
&= \frac{3}{2}(3 + \sqrt{5}) - 1 \\
&= \frac{1}{2}(7 + 3\sqrt{5}) \\
&= \frac{1}{4}(14 + 6\sqrt{5}) \\
&= \left[\frac{1}{2}(3 + \sqrt{5}) \right]^2 \\
\tan \theta &= \frac{1}{2}(3 + \sqrt{5})
\end{aligned}$$

The regular icosahedron can be regarded as 20 identical right pyramids with equilateral triangular base, side = a . The volume of the regular icosahedron is

$$\begin{aligned}
20 \times \frac{1}{3} \times \frac{1}{2} \times a^2 \sin 60^\circ \times PQ &= \frac{5a^2}{\sqrt{3}} \times RQ \tan \theta \\
&= \frac{5a^2}{\sqrt{3}} \times \frac{a}{2\sqrt{3}} \times \frac{1}{2}(3 + \sqrt{5}) \\
&= \frac{5}{12}(3 + \sqrt{5})a^3
\end{aligned}$$

參考資料：等周問題 香港時代圖書有限公司，1975 年