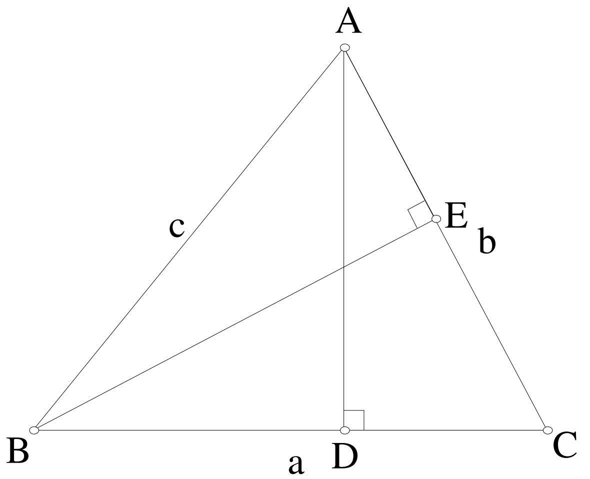
Sine formula

Created by Mr. Francis Hung Last updated: August 15, 2022

In ΔABC, let AD, BE, CF be the altitudes.

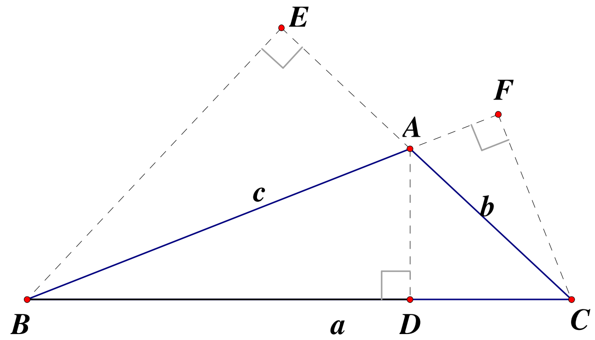
If ΔABC is an acute-angled triangle, then
$AD = c \sin B = b \sin C$
$BE = a \sin C = c \sin A$



$$\Rightarrow \begin{cases} \dfrac{b}{\sin B} = \dfrac{c}{\sin C} \\ \dfrac{a}{\sin A} = \dfrac{c}{\sin C} \end{cases}$$

$$\therefore \ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

If ΔABC is an obtuse-angled triangle,
WLOG assume $\angle A > 90°$, then
$AD = c \sin B = b \sin C$
$BE = a \sin C = c \sin (180° - A)$



$$\Rightarrow \begin{cases} \dfrac{b}{\sin B} = \dfrac{c}{\sin C} \\ \dfrac{a}{\sin A} = \dfrac{c}{\sin C} \end{cases}$$

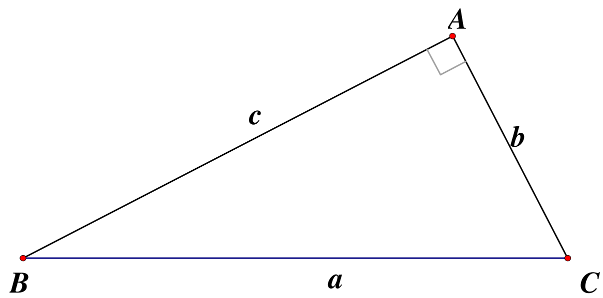
$$\therefore \ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

If ΔABC is a right-angled triangle,
WLOG assume $\angle A = 90°$, then

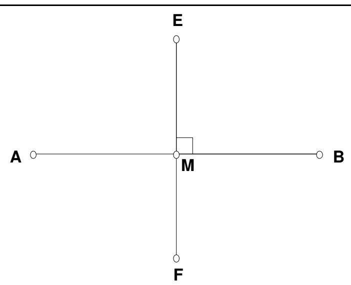
$$b = a \sin B \Rightarrow \frac{a}{\sin A} = a = \frac{b}{\sin B}$$

$$c = a \sin C \Rightarrow \frac{a}{\sin A} = a = \frac{c}{\sin C}$$

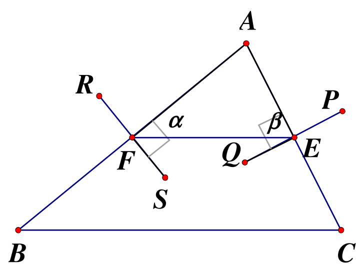
$$\therefore \ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The **perpendicular bisector** of a line segment.
Given a line segment AB.
A line segment EF intersects AB at M.
If (1) $EF \perp AB$ and (2) $AM = MB$;
then EF is called the perpendicular bisector of AB.



Theorem 1 Given ΔABC. The perpendicular bisectors AB and AC will intersect at a point O.
Let PQ and RS be the perpendicular bisectors of AC and AB respectively. E and F are the mid-points of AC and AB respectively.
Join EF. Let $\angle AFE = \alpha$, $\angle AEF = \beta$
$\angle SFE = 90° - \alpha$, $\angle QEF = 90° - \beta$
$\angle SFE + \angle QEF = 90° - \alpha + 90° - \beta = 180° - (\alpha + \beta) < 180°$
$\therefore$ PQ and RS intersect at a point O.



Circumcircle and Sine formula

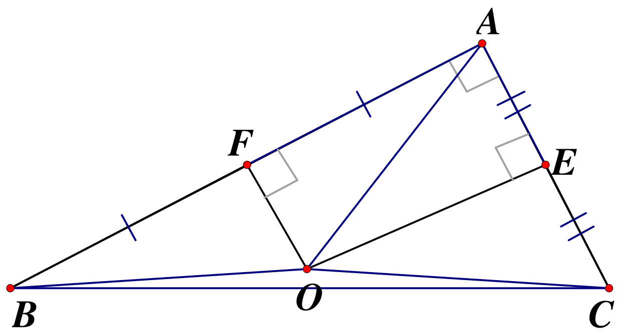
Theorem 2 $\angle A$ is the largest angle in ΔABC. The perpendicular bisectors AB and AC intersect at O.

Case 1 If $\angle A = 90°$, then O is the mid-point of BC.

Case 2 If $\angle A < 90°$, then O lies inside ΔABC.

Case 3 If $\angle A > 90°$, then O lies outside ΔABC.

Proof: Case 1 If $\angle A = 90°$.



Let FO and EO be the perpendicular bisectors of AB and AC respectively which meet at O. E and F are the mid-points of AC and AB respectively.

Join AO, BO and CO.

$AEOF$ is a rectangle. (It has 3 right angles)

$\angle FOE = 90°$ ($\angle$ sum of polygon)

$\Delta BOF \cong \Delta AOF$ (S.A.S.)

$\Delta AOF \cong \Delta OAE$ (S.S.S.)

$\Delta OAE \cong \Delta OCE$ (S.A.S.)

$\therefore \Delta BOF \cong \Delta OCE$

$\angle FBO = \angle EOC$ (corr. $\angle$s, $\cong \Delta$s)

$\angle BOF = \angle OCE$ (corr. $\angle$s, $\cong \Delta$s)

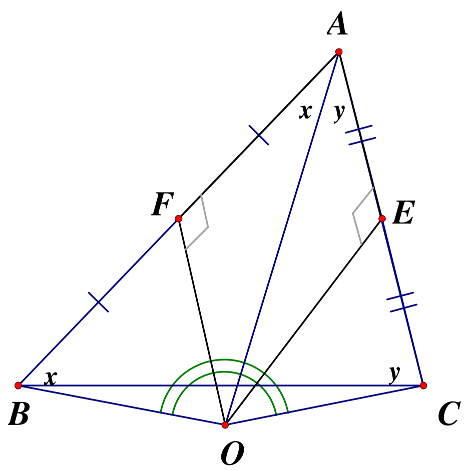
In ΔBOF, $\angle BOF + \angle FBO = 90°$ ($\angle$ sum of Δ)

$\therefore \angle BOF + \angle FOE + \angle EOC = 180°$

B, O, C are collinear

The two perpendicular bisectors intersect at O, which is the mid-point of BC.

Case 2 If $\angle A < 90°$.



Join AO, BO and CO.

With the same arguments as above,

$\Delta BOF \cong \Delta AOF$, $\Delta OAE \cong \Delta OCE$ (S.A.S.)

$OB = OA = OC$ (corr. sides, $\cong \Delta$s)

Let $\angle OAF = x$, $\angle OAE = y$.

Then $\angle B, \angle C \le \angle A = x + y < 90°$

$\angle OBF = x$, $\angle OCE = y$ (corr. $\angle$s, $\cong \Delta$s)

$\angle AOF = \angle BOF = 90° - x$ ($\angle$ sum of Δ)

$\angle AOE = \angle COE = 90° - y$ ($\angle$ sum of Δ)

Consider the marked angle $\angle BOC$ in the figure.

$\angle BOC = 2(90° - x) + 2(90° - y)$

$= 360° - 2(x + y)$

$> 360° - 2\times 90° > 180°$ …… (*)

If O lies on BC, then $\angle BOC = 180°$

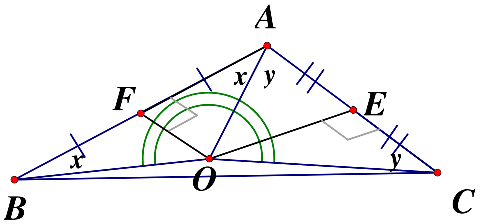
If O lies outsides ΔABC, then $\angle BOC < 180°$

Both cases are contradictory to (*)

$\therefore$ O must lie inside ΔABC.

Circumcircle and Sine formula

Case 3 If $\angle A > 90°$.
Join AO, BO and CO.
With the same arguments as above,
$\Delta BOF \cong \Delta AOF$, $\Delta OAE \cong \Delta OCE$ (S.A.S.)
$OB = OA = OC$ (corr. sides, $\cong\Delta$s)
Let $\angle OAF = x$, $\angle OAE = y$.
Then $\angle A = x + y > 90° > \angle B, \angle C$
$\angle OBF = x$, $\angle OCE = y$ (corr. $\angle$s, $\cong\Delta$s)
$\angle AOF = \angle BOF = 90° - x$ ($\angle$ sum of Δ)
$\angle AOE = \angle COE = 90° - y$ ($\angle$ sum of Δ)



Consider the marked reflex angle $\angle BOC$ in the figure.

$$\begin{aligned}\text{Reflex } \angle BOC &= 2(90° - x) + 2(90° - y)\\ &= 360° - 2(x + y)\\ &< 360° - 2\times 90° = 180° \cdots\cdots (**)\end{aligned}$$

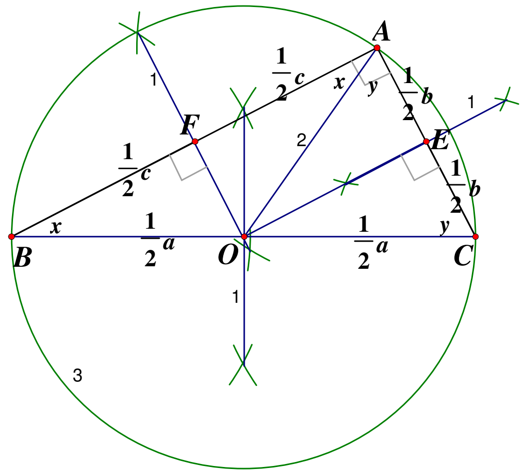
If O lies on BC, then $\angle BOC = 180°$
If O lies insides ΔABC, then reflex $\angle BOC > 180°$
Both cases are contradictory to (**)
$\therefore$ O must lie outside ΔABC.

Theorem 3 The three perpendicular bisectors of a triangle ABC are concurrent at a point O.

O is called the **circumscribed centre** (or **circumcentre** in short form). We can use O as centre to draw a circle to pass through A, B, C. The circle is called the **circumscribed circle** (or **circum-circle** in short form) and the radius (R) is called the **circum-radius**.

Furthermore, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$.

Case 1 ΔABC is a right-angled triangle. WLOG assume $\angle A = 90°$.
(1) Draw the perpendicular bisector of AB and the perpendicular bisector of AC.
By **Theorem 2** case 1, the three perpendicular bisectors are concurrent at the mid-point of BC, O.
(2) Join OA.
(3) Use O as centre, OA as radius to draw a circle.
BC is the diameter of the circle.



$\angle B = \dfrac{1}{2}\angle AOC = \angle COE$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$\angle C = \dfrac{1}{2}\angle AOB = \angle BOF$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

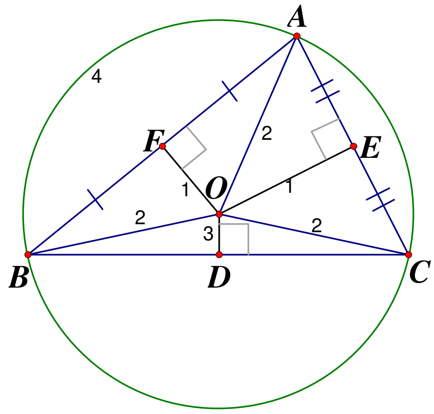
In ΔCOE, $\dfrac{CE}{OC} = \sin\angle COE \Rightarrow \dfrac{b}{2R} = \sin B \Rightarrow \dfrac{b}{\sin B} = 2R$

In ΔBOF, $\dfrac{BF}{OB} = \sin\angle BOF \Rightarrow \dfrac{c}{2R} = \sin C \Rightarrow \dfrac{c}{\sin C} = 2R$

$\dfrac{a}{\sin A} = \dfrac{2R}{\sin 90^\circ} = 2R$

Circumcircle and Sine formula

Case 2 ΔABC is an acute-angled triangle. $\angle A \geq \angle B, \angle C$



(1) Let OE and OF be the two perpendicular bisectors of AC and AB respectively which intersect at O.
(2) Join OA, OB and OC.
(3) Draw $OD \perp BC$, where D is the foot of perpendicular.

By the property of perpendicular bisectors,
$AE = CE$, $AF = BF$, $OE \perp AC$, $OF \perp AB$.
Then $\Delta AOE \cong \Delta COE$ (S.A.S.)
$\Delta AOF \cong \Delta BOF$ (S.A.S.)
$\therefore OB = OA = OC$ (corr. sides, $\cong \Delta$s)
$\Delta BOD \cong \Delta COD$ (R.H.S.)
$\therefore BD = CD$ (corr. sides, $\cong \Delta$s)
$\therefore OD$ is a perpendicular bisector of BC.
The three perpendicular bisectors are concurrent at a point O.

(4) Use O as centre and OA as radius to draw a circle which passes through ΔABC.

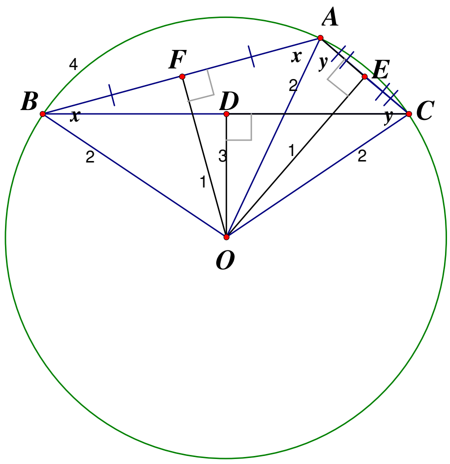
$\therefore \angle BOD = \angle COD = \angle A$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$) and $BD = \frac{a}{2}$ (corr. sides, $\cong \Delta$s)

Let the circumscribed radius be R.

In ΔBOD, $\frac{BD}{OB} = \sin \angle BOD \Rightarrow \frac{\frac{a}{2}}{R} = \sin A \Rightarrow \frac{a}{\sin A} = 2R$

$\angle AOC = 2\angle B$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)	$\angle AOB = 2\angle C$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\because \Delta AOE \cong \Delta COE$	$\because \Delta AOF \cong \Delta BOF$
$\therefore \angle AOE = \angle COE = \angle B$ and $CE = \frac{b}{2}$	$\therefore \angle AOF = \angle BOF = \angle C$ and $BF = \frac{c}{2}$
In ΔCOE, $\frac{CE}{OC} = \sin \angle COE$	In ΔBOF, $\frac{BF}{OB} = \sin \angle BOF$
$\Rightarrow \frac{\frac{b}{2}}{R} = \sin B \Rightarrow \frac{b}{\sin B} = 2R$	$\Rightarrow \frac{\frac{c}{2}}{R} = \sin C \Rightarrow \frac{c}{\sin C} = 2R$

Case 3 ΔABC is an obtuse-angled triangle. WLOG assume $\angle A > 90° > \angle B, \angle C$



(1) Let OE and OF be the two perpendicular bisectors of AC and AB respectively which intersect at O.
(2) Join OA, OB and OC.
(3) Draw $OD \perp BC$, where D is the foot of perpendicular.

By the property of perpendicular bisectors,
$AE = CE$, $AF = BF$, $OE \perp AC$, $OF \perp AB$.
Then $\Delta AOE \cong \Delta COE$ (S.A.S.)
$\Delta AOF \cong \Delta BOF$ (S.A.S.)
$\therefore OB = OA = OC$ (corr. sides, $\cong \Delta$s)
$\Delta BOD \cong \Delta COD$ (R.H.S.)
$\therefore BD = CD$ (corr. sides, $\cong \Delta$s)
$\therefore OD$ is a perpendicular bisector of BC.
The three perpendicular bisectors are concurrent at a point O.

(4) Use O as centre and OA as radius to draw a circle which passes through ΔABC.

Reflex $\angle BOC = 2\angle A$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\angle BOC = 360° - 2\angle A$ ($\angle$s at a point)
$\because \Delta BOD \cong \Delta COD$

$\therefore \angle BOD = \angle COD = 180° - \angle A$ (corr. $\angle$s, $\cong \Delta$'s) and $BD = \frac{a}{2}$ (corr. sides, $\cong \Delta$s)

Circumcircle and Sine formula

Let the circumscribed radius be R.

In ΔBOD, $\dfrac{BD}{OB} = \sin\angle BOD$

$\dfrac{\frac{a}{2}}{R} = \sin(180^\circ - A)$

$\dfrac{a}{\sin A} = 2R$

$\angle AOC = 2\angle B$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)	$\angle AOB = 2\angle C$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)
$\because\ \Delta AOE \cong \Delta COE$	$\because\ \Delta AOF \cong \Delta BOF$
$\therefore\ \angle AOE = \angle COE = \angle B$ and $CE = \dfrac{b}{2}$	$\therefore\ \angle AOF = \angle BOF = \angle C$ and $BF = \dfrac{c}{2}$
In ΔCOE, $\dfrac{CE}{OC} = \sin\angle COE$	In ΔBOF, $\dfrac{BF}{OB} = \sin\angle BOF$
$\dfrac{\frac{b}{2}}{R} = \sin B$	$\dfrac{\frac{c}{2}}{R} = \sin C$
$\dfrac{b}{\sin B} = 2R$	$\dfrac{c}{\sin C} = 2R$

Therefore, we have proved the **Sine formula** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$,
where R is the radius of the circumscribed circle.

Use Heron's formula to find the area of $\Delta ABC = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = \dfrac{1}{2}(a+b+c)$ (Half of the perimeter of ΔABC).

$\Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = \dfrac{1}{2}ab\sin C$

$\Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = \dfrac{1}{2}ab\dfrac{c}{2R}$

$\Rightarrow R = \dfrac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$

This is the formula of the radius in terms of the sides of triangle.

Example (A.S.A.) $a = 5$, $\angle B = 60^\circ$, $\angle C = 45^\circ$.Find b.

$\angle A = 75^\circ$ ($\angle$ sum of Δ)

$\dfrac{5}{\sin 75^\circ} = \dfrac{b}{\sin 60^\circ}$

$b = \dfrac{5\sin 60^\circ}{\sin 75^\circ}$ = 4.48 (correct to 3 sig. fig.)

Example (S.S.A.) $a = 5$, $b = 6$, $\angle B = 60^\circ$. Find $\angle A$.

$\dfrac{5}{\sin A} = \dfrac{6}{\sin 60^\circ}$

$\sin A = 0.721687836$

$A = 46.2^\circ$ or $180^\circ - 46.2^\circ$

$A = 46.2^\circ$ or 133.8°

But when $A = 133.8^\circ$, $A + B = 193.8^\circ > 180^\circ$

$\therefore\ A = 46.2^\circ$ only.