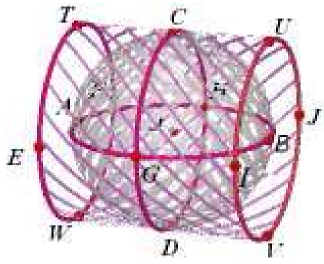
The circumscribing cylinder

Reference: Advance Level Pure Mathematics by S.L.Green p.187-189

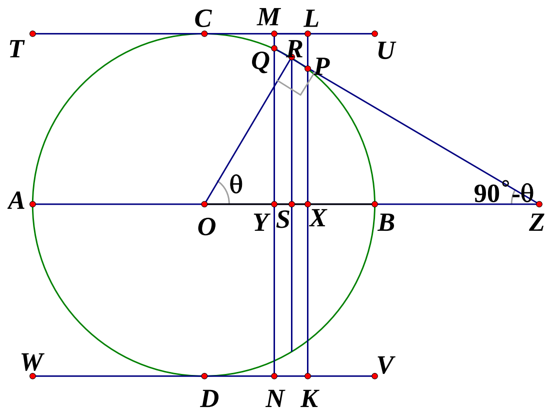
Created by Mr. Francis Hung. Last updated: 04 February 2016



The figure shows a right cylinder circumscribing a sphere. The base radius of the cylinder is the same as that of the sphere. The height of the cylinder is the same as the diameter of the sphere. *TCU* and *WDV* are the heights of the cylinder. *TW* and *EF* are perpendicular diameters of the left base of the cylinder, with centre at *A*. *IJ* and *UV* are perpendicular diameters of the right base of the cylinder, with centre at *B*. *O* is the centre of the sphere. *CD* and *GH* are perpendicular diameters of the sphere and the cylinder.

Let the radius of the sphere be r, then $OC = OD = BU = BV = AT = AW = r$.

We are going to show that the curved surface area of the cylinder is equal to the surface area of the sphere.



The figure shows the cross section view of the circumscribing cylinder. The circle centred at *O*. *TU* and *WV* are the heights of the cylinder which touch the sphere at *C* and *D* respectively. Two planes *LXK* and *MYN* , parallel to the bases of the cylinder, cut the sphere at *P* and *Q* respectively. *LK* and *MN* are the diameters of the cylinder. *LK* and *MN* cut the circle at *P* and *Q* respectively, they also cut *AB* at *Y* and *X* respectively. Suppose the mid-points of *PQ* and the mid-point of *XY* are *R* and *S* respectively.

The distance between *P* and *Q* are so small that we may assume that *R* lies on the circle.

Claim: The curved surface area of the ring generated by revolving *PQ* about the axis *AB* is equal to the curved surface area of the ring generated by revolving *ML* about the axis *AB*.

Proof: Let the radius of the sphere be r.

The curved surface of the ring generated by revolving $\overset{\frown}{PQ}$ about the axis *AB* can be regarded as the curved surface of a frustum. Join *OR*. The tangent at *R* meets *AB* produced at *Z*.

Let $\angle ROZ = \theta$, $\angle ORZ = 90°$ (tangent ⊥ radius), $\angle OZR = 90° - \theta$ (∠s sum of Δ)

$RS = r \sin\theta$

The curved surface area of the frustum $= 2\pi\, RS\cdot PQ = 2\pi r \sin\theta\cdot PQ$ ⋯⋯ (1)

In ΔPXZ, $\angle ZPX = \theta$ (∠s sum of Δ)

$\angle QPL = \theta$ (vert. opp. ∠s)

$ML = PQ \sin\theta$

The curved surface area of the ring generated by revolving *ML* about the axis *AB*

$= 2\pi r\cdot ML = 2\pi r\cdot PQ\sin\theta$ ⋯⋯ (2)

∴ (1) = (2), and the claim is proved.

If we cut the cylinder and the sphere into infinite slices perpendicular to *AB*, then sum up each of the corresponding equal areas. So we have proved that the curved surface area of a sphere and that of the circumscribed cylinder are equal.

Let S be the surface area of the sphere. Then $S = 2\pi r\cdot(2r) = 4\pi r^2$