Volume of a frustum

Created by Mr. Francis Hung. Last updated: 15 February 2016

In the figure 1(*a*), *PABCD* is a right pyramid with square base *ABCD*, length of side = *b*.

P is the vertex. *O* is the projection of *P* on *ABCD*.

The height *PO* = *h* and $\angle POC = 90°$.

Volume $= \frac{1}{3} r^2 h$

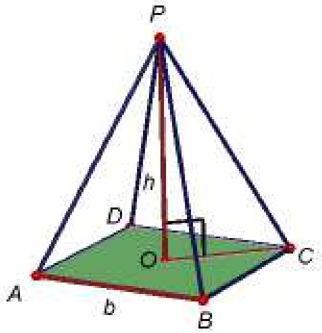

Figure 1 (*a*)

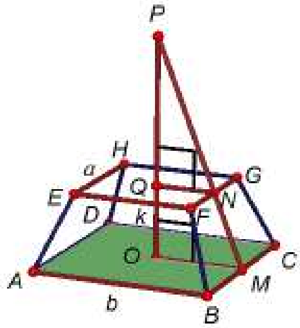

Figure 1 (*b*)

In the figure 1(*b*), *ABCDHEFG* is a frustum formed by cutting off a smaller pyramid *PEFGH* with length of side = *a* from the larger pyramid *PABCD*.

Q is the projection of *P* on *EFGH*.

Let *M* and *N* be the mid-points of *BC* and *FG* respectively.

$\angle PQN = 90°$, $\angle POM = 90°$.

Let the distance between the upper base *EFGH* and the lower base *ABCD* be *k*, i.e. *QO* = *k*.

Then the volume of the frustum is $V = \frac{k}{3}\left(a^2 + ab + b^2\right)$.

Proof: $\Delta PQN \sim \Delta POM$ (A.A.A.)

$$\frac{PO}{PQ} = \frac{OM}{QN} \quad \text{(corr. sides, ~}\Delta\text{s)}$$

$$\frac{PQ+k}{PQ} = \frac{\frac{b}{2}}{\frac{a}{2}}$$

$$1 + \frac{k}{PQ} = \frac{b}{a}$$

$$\frac{k}{PQ} = \frac{b}{a} - 1 = \frac{b-a}{a}$$

$$PQ = \frac{ka}{b-a}$$

V = volume of the big pyramid – volume of the small pyramid

$$= \frac{1}{3}b^2\left(\frac{ka}{b-a} + k\right) - \frac{1}{3}a^2 \cdot \frac{ka}{b-a}$$

$$= \frac{1}{3}b^2 \cdot \frac{kb}{b-a} - \frac{1}{3}a^2 \cdot \frac{ka}{b-a}$$

$$= \frac{k}{3(b-a)} \cdot \left(b^3 - a^3\right)$$

$$= \frac{k}{3(b-a)} \cdot (b-a)\left(b^2 + ab + a^2\right)$$

$$= \frac{k}{3}\left(a^2 + ab + b^2\right)$$

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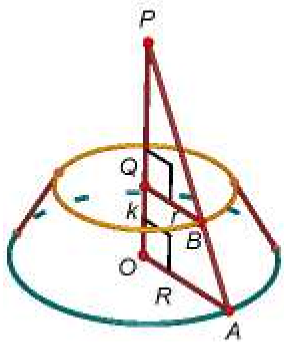

Figure 2

In the figure 2, a right circular frustum is formed by cutting off a smaller cone with radius r from a larger cone with radius R.

P is the common vertex of the cones. O and Q are the projection of P on the top circle and the base circle respectively. $\angle PQB = 90°$, $\angle POA = 90°$.

PBA is a straight line, $QB = r$ and $OA = R$.

The height of the frustum $OQ = k$.

To find the volume and the curved surface area of the frustum.

$\Delta PQB \sim \Delta POA$ (A.A.A.)

$\frac{PO}{PQ} = \frac{OA}{QB}$ (corr. sides, ~Δs)

$$\frac{PQ+k}{PQ} = \frac{R}{r}$$

$$1+\frac{k}{PQ} = \frac{R}{r}$$

$$\frac{k}{PQ} = \frac{R}{r} - 1 = \frac{R-r}{r}$$

$$PQ = \frac{kr}{R-r}$$

V = volume of the big cone – volume of the small cone

$$= \frac{\pi}{3}R^2\left(\frac{kr}{R-r}+k\right) - \frac{\pi}{3}r^2 \cdot \frac{kr}{R-r}$$

$$= \frac{\pi}{3}R^2 \cdot \frac{kR}{R-r} - \frac{\pi}{3}r^2 \cdot \frac{kr}{R-r}$$

$$= \frac{\pi k}{3(R-r)} \cdot \left(R^3 - r^3\right)$$

$$= \frac{\pi k}{3(R-r)} \cdot (R-r)\left(R^2 + rR + r^2\right)$$

$$= \frac{\pi k}{3}\left(R^2 + rR + r^2\right)$$

S = curved surface area of big cone – curved surface area of the small cone

$$= \pi R \cdot AP - \pi r \cdot BP$$

$$= \pi R \cdot \sqrt{R^2 + OP^2} - \pi r \cdot \sqrt{r^2 + PQ^2}$$

$$= \pi R \cdot \sqrt{R^2 + \left(\frac{kr}{R-r}+k\right)^2} - \pi r \cdot \sqrt{r^2 + \left(\frac{kr}{R-r}\right)^2}$$

$$= \pi R \cdot \sqrt{R^2 + \left(\frac{kR}{R-r}\right)^2} - \pi r \cdot \sqrt{r^2 + \left(\frac{kr}{R-r}\right)^2}$$

$$= \frac{\pi R^2}{R-r} \cdot \sqrt{(R-r)^2 + k^2} - \frac{\pi r^2}{R-r} \cdot \sqrt{(R-r)^2 + k^2}$$

$$= \frac{\pi\left(R^2 - r^2\right)}{R-r} \cdot \sqrt{(R-r)^2 + k^2}$$

$$= \pi(R + r)AB$$

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In the figure 3, a frustum is formed by cutting off a smaller triangular pyramid from a larger triangular pyramid.

P is the vertex of the triangular pyramid.

ABC and *DEF* are the upper bases and the lower bases respectively such that $AB // DE$, $BC // EF$, $AC // DF$.

O and *Q* are the projection of *P* on *ABC* and *DEF* respectively. $\angle PQF = 90°$, $\angle POC = 90°$. *P*, *F*, *C* are collinear, $BC = b$ and $EF = a$.

The height of the frustum $OQ = k$.

To find the volume of the frustum *V* in terms of *k*, *a* and *b*.

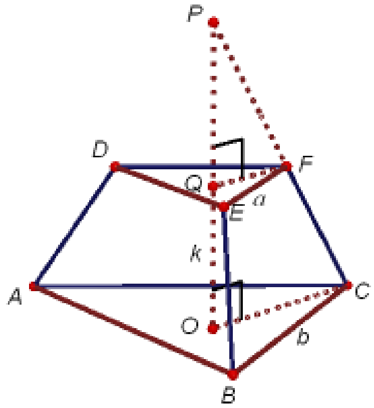

Figure 3

$\Delta PEF \sim \Delta PBC$, $\Delta PQF \sim \Delta POC$, $\Delta DEF \sim \Delta ABC$ (A.A.A.)

$$\frac{PO}{PQ}=\frac{PC}{PF}=\frac{BC}{EF}=\frac{AB}{DE} \quad \text{(corr. sides, ~}\Delta\text{s)}$$

$$\frac{PQ+k}{PQ}=\frac{b}{a}$$

$$1+\frac{k}{PQ}=\frac{b}{a}$$

$$\frac{k}{PQ}=\frac{b}{a}-1=\frac{b-a}{a}$$

$$PQ=\frac{ka}{b-a}$$

V = volume of the big pyramid – volume of the small pyramid

$$=\frac{1}{3}\cdot\frac{1}{2}AB\cdot BC\sin\angle ABC\left(\frac{ka}{b-a}+k\right)-\frac{1}{3}\cdot\frac{1}{2}DE\cdot EF\sin\angle DEF\cdot\frac{ka}{b-a}$$

$$=\frac{1}{3}\cdot\frac{1}{2}bd\sin\theta\cdot\frac{kb}{b-a}-\frac{1}{3}\cdot\frac{1}{2}ac\sin\theta\cdot\frac{ka}{b-a}$$, where $\theta = \angle ABC = \angle DEF$, $DE = b$, $AB = d$

$$=\frac{k\sin\theta}{6(b-a)}\cdot\left(b^2d-a^2c\right)=\frac{kt\sin\theta}{6(b-a)}\cdot\left(b^3-a^3\right)$$, let $c = at$, $d = bt$

$$=\frac{kt\sin\theta}{6}\left(a^2+ab+b^2\right)$$

$$=\frac{k}{3}\left(\frac{1}{2}a\cdot at\sin\theta+\frac{1}{2}abt\sin\theta+\frac{1}{2}b\cdot bt\sin\theta\right)$$

$$=\frac{k}{3}\left(\frac{1}{2}a\cdot at\sin\theta+\sqrt{\frac{1}{2}a\cdot at\sin\theta\cdot\frac{1}{2}b\cdot bt\sin\theta}+\frac{1}{2}b\cdot bt\sin\theta\right)$$

$$=\frac{k}{3}\left(B_1+\sqrt{B_1\cdot B_2}+B_2\right)$$, where B_1 = area of ΔABC and B_2 = area of ΔDEF respectively.