Given the volume of a frustum, to find the surface area and the volume of a sphere

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Cavalier's Principle:

Given two solids have the same heights. If the cross section areas of the two solids of the same height are equal, then these two solids have the same volumes.

The figures on the right are illustration of Cavalier's Principle. The two solids in Figure 1(a) and Figure 1(b) have the same heights and same base radii.

Figure 1(b) is a right cylinder. Figure 1(a) consists of several smaller cylinders and the line joining the centres of each of the base circle of the smaller cylinders is NOT perpendicular to the base.

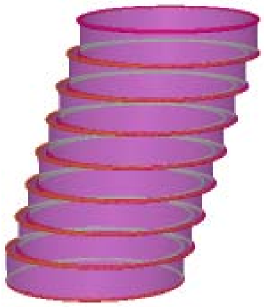

Figure 1(a)

Figure 1(b)

Since the cross section of the same height of both solids are identical, they have the same volumes.

(1) **The volume of a sphere**

In the figure 2(a), $ABCD$ is a right solid cylinder. P and Q are the centres of the top and the base respectively. AB and CD are the diameters of the top and the base respectively. C is vertically below A and D is vertically below B. $PA = QC = R$ (radius)

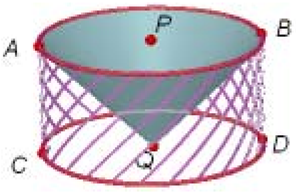

Figure 2(a)

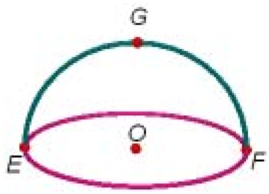

Figure 2(b)

$AC = BD = R$ (height of the circular)

A hole in the shape of an right inverted cone is scooped out of the cylinder.

The volume of the solid in Figure 2(a) = volume of the cylinder – volume of the inverted cone

$$= \pi R^2 \cdot R - \frac{\pi}{3} R^2 \cdot R = \frac{2\pi}{3} R^3$$

Figure 2(b) shows a hemi-sphere, centre at O, radius R. We want to show that the volume in Figure 2(a) is equal to the volume of a hemi-sphere.

Figure 3(a) shows the front view (cross section) of the cylinder with a hole being scooped off.

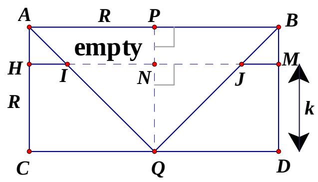

Figure 3(a)

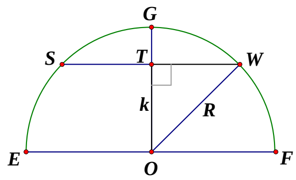

Figure 3(b)

A plane $HINJM$ parallel to the base, and at a distance k units from the base is cut through the cylinder. The top view of the cutting plane $HINJM$ is shown in Figure 4(a) which is in the form of a ring. Figure 3(b) shows the front view (cross section) of the sphere. A plane STW parallel to the base of the hemisphere, and at a distance k units from the base is cut through the sphere. The top view of the cutting plane STW is shown in Figure 4(b) which is in the form of a circle. In Figure 3(a), $\Delta APQ \sim \Delta INQ$ (A.A.A.)

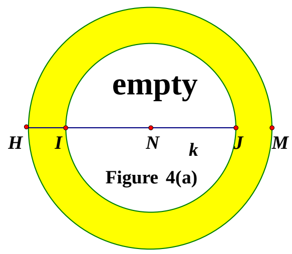

Figure 4(a)

Top views

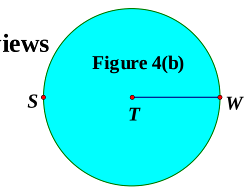

Figure 4(b)

$IN : NQ = AP : PQ$ (corr. sides, ~Δs)

$IN : k = R : R = 1 = 1 \Rightarrow IN = k$

Area of the ring in Figure 4(a) = $\pi(R^2 - k^2)$

In Figure 3(b), $TW^2 = R^2 - k^2$ (Pythagoras' theorem)

Area of the circle in Figure 4(b) = $\pi(R^2 - k^2)$ = area of the ring in Figure 4(a)

∵ The solids in Figure 2(a) and the hemisphere in Figure 2(b) have the same height and the same cross section area ∴ By Cavalier's Principle, the two solids have the same volume.

Volume of hemisphere $= \frac{2\pi}{3} R^3$

$\Rightarrow$ volume of a sphere $= \frac{4\pi}{3} R^3$

(2) Surface area of a sphere

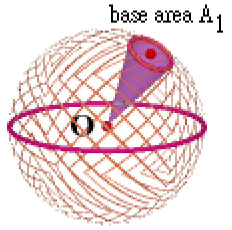

Figure 5

Figure 5 shows a sphere centre at O, with radius R.
We want to find the surface area of the sphere, let it be S.
A tiny hole, in the shape of inverted cone, with base area A_1 and height R is drilled from the surface of the sphere to the centre of the sphere.

Volume of the cone $= \frac{1}{3} \cdot A_1 R$

Another tiny hole, in the shape of inverted cone, with base area A_2 and height R is drilled from the surface of the sphere to the centre of the sphere.

Volume of the second cone $= \frac{1}{3} \cdot A_2 R$

The process is repeated until the whole sphere is scooped off.
Volume of the sphere = total volumes of the tiny cone being scooped off

$$= \frac{1}{3} \cdot A_1 R + \frac{1}{3} \cdot A_2 R + \cdots\cdots$$

$$\frac{4\pi R^3}{3} = \frac{R}{3} \cdot \left(A_1 + A_2 + \cdots\cdots\right)$$

On the other hand, $A_1 + A_2 + \cdots\cdots$ = surface area of a sphere = S

$$\therefore \quad \frac{4\pi R^3}{3} = \frac{R}{3} \cdot S$$

$S = 4\pi R^2$