Limit concepts

Created by Mr. Francis Hung on 20220211. Last updated: 11 February 2022.

Objective: to teach the concept of limits through examples.

Introduction of limit concepts **(1 minute)**

(1) $\lim_{x\to a} f(x)$ is read as limit as x tends to a of f(x).

(2) If the graph of y = f(x) is continuous at $x = a$, then the value of $\lim_{x\to a} f(x)$ is f(a).

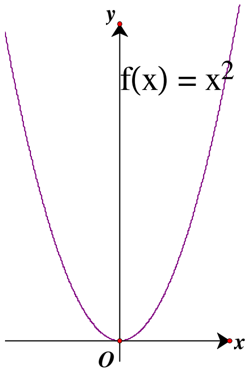
We write $\lim_{x\to a} f(x) = f(a)$.

Example 1 (3 minutes)

For the graph of $y = x^2$, which is continuous everywhere.

We write f(x) = x^2, and $\lim_{x\to 2} x^2 = 2^2 = 4$

What is the value of $\lim_{x\to -1} x^2$?



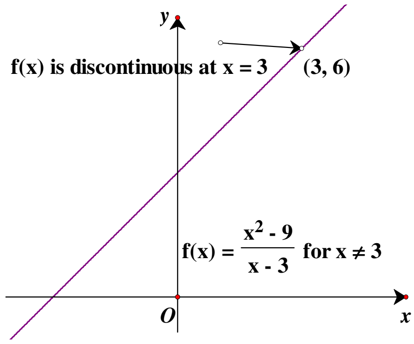
Example 2 (8 minutes)

Define a function $f(x) = \frac{x^2-9}{x-3}$ for $x \neq 3$.

Then $f(x) = \frac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$.

$$f(x) = \begin{cases} x+3 & \text{for } x \neq 3 \\ \text{undefined} & \text{for } x = 3 \end{cases}$$

The graph of y = f(x) is the straight line $y = x + 3$ except $x = 3$, which is discontinuous at $x = 3$, as shown on the right.



As y = f(x) is continuous at $x = 2$, then we have $\lim_{x\to 2} \frac{x^2-9}{x-3} = f(2) = 2 + 3 = 5$

What is the value of $\lim_{x\to -1} \frac{x^2-9}{x-3}$?

As y = f(x) is discontinuous at $x = 3$, then we consider the **left-hand limit** and **the right-hand limit**.

Left-hand limit is to find the limiting value of y = f(x) as x tends to a (in this case x tends to 3) from the left side of a. x is very close to a and smaller than a but **not equal** to a.

Example of left-hand limit:

x	2.5	2.6	2.88	2.932	2.997	2.9999	2.999989
f(x)	5.5	5.6	5.88	5.932	5.997	5.9999	5.999989

We see that $\lim_{x\to 3^-} \frac{x^2-9}{x-3} = 6$ (the left-hand limit is 6.)

Similarly, the right-hand limit to find the limiting value of y = f(x) as x tends to a (in this case x tends to 3) from the right side of a. x is very close to a and bigger than a but **not equal** to a.

Example of right-hand limit:

x	4.2	3.55	3.021	3.0078	3.00002	3.0000004
f(x)	7.2	6.55	6.021	6.0078	6.00002	6.0000004

We see that $\lim_{x\to 3^+} \frac{x^2-9}{x-3} = 6$ (the right-hand limit is 6.)

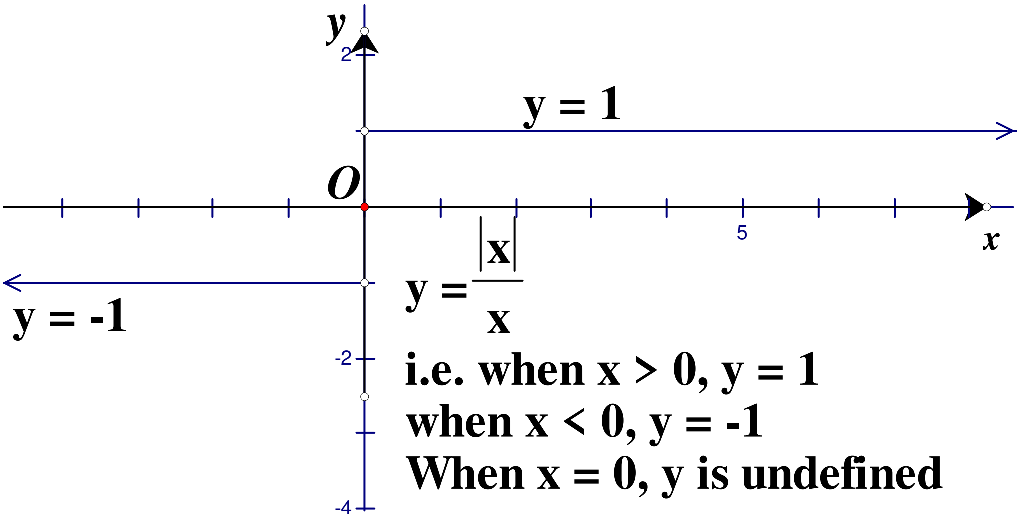
Theorem The limit of a function exists if and only if the left hand limit is equal to the right limit.

In the above case, $\because \lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^+} \frac{x^2-9}{x-3} = 6$

$\therefore \lim_{x\to 3} \frac{x^2-9}{x-3}$ exists and equal to 6. (We write $\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$)

Example 3 (3 minutes)

Consider the graph of $y=\frac{|x|}{x}=\begin{cases}1 & \text{when } x>0\\ \text{undefined} & \text{when } x=0\\ -1 & \text{when } x<0\end{cases}$ consists of two rays and is discontinuous at $x = 0$.



At $x = 4.5$, (the graph is continuous at $x = 4.5$), $\lim_{x\to 4.5}\frac{|x|}{x}$

$\lim_{x\to 0^-}\frac{|x|}{x}=-1$ (find the limit of the function $\frac{|x|}{x}$ as x tends to 0 from the left hand side.)

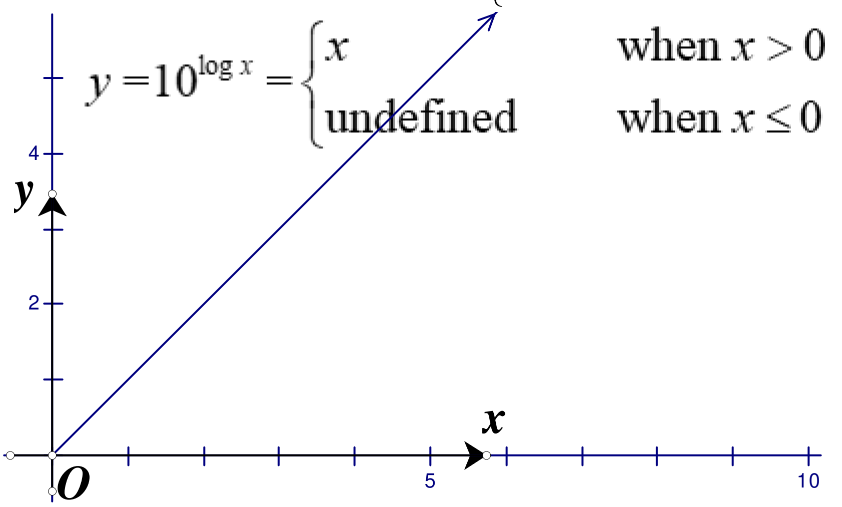
$\lim_{x\to 0^+}\frac{|x|}{x}=1$ (find the limit of the function $\frac{|x|}{x}$ as x tends to 0 from the right hand side.)

$\because \lim_{x\to 0^-}\frac{|x|}{x}\neq\lim_{x\to 0^+}\frac{|x|}{x}$

$\therefore \lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Example 4 (3 minutes)

Consider a new function $y=10^{\log x}=\begin{cases}x & \text{when } x>0\\ \text{undefined} & \text{when } x\le 0\end{cases}$



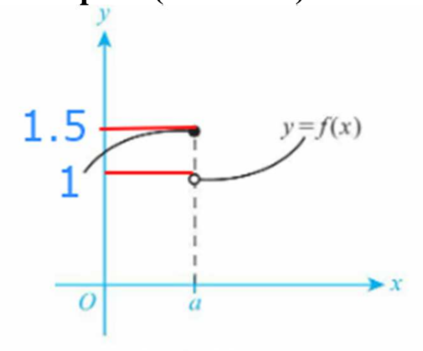
$\lim_{x\to 9.99}10^{\log x}=9.99$, $\lim_{x\to -3}10^{\log x}$ does not exists,

$\lim_{x\to 0^-}10^{\log x}$ does not exists , $\lim_{x\to 0^+}10^{\log x}=0$

$\because \lim_{x\to 0^-}10^{\log x}\neq\lim_{x\to 0^+}10^{\log x}$

$\therefore \lim_{x\to 0}\left(10^{\log x}\right)$ does not exist

Example 5 (3 minutes)



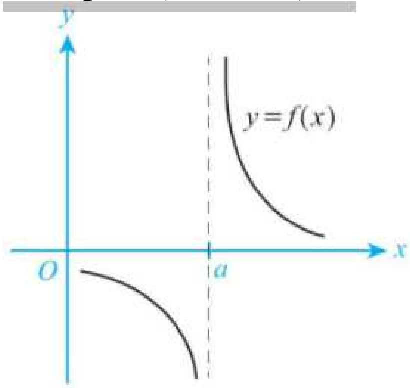
$f(a)$ is defined, and

$\lim_{x\to a^-} f(x) = 1.5$, $\lim_{x\to a^+} f(x) = 1$

$\because$ $\lim_{x\to a^-} f(x) \ne \lim_{x\to a^+} f(x)$

$\therefore$ $\lim_{x\to a} f(x)$ does not exist

Example 6 (3 minutes)



$f(a)$ is undefined, and

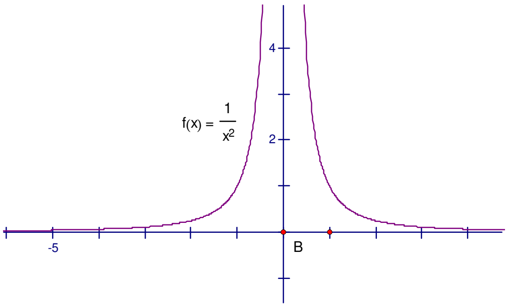
$\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = \infty$

$\because$ $\lim_{x\to a^-} f(x) \ne \lim_{x\to a^+} f(x)$

$\therefore$ $\lim_{x\to a} f(x)$ does not exist

Example 7 (3 minutes)

Consider a function $y = \frac{1}{x^2}$ for $x \ne 0$



When $x = 0$, y is undefined.

The graph is discontinuous at $x = 0$.

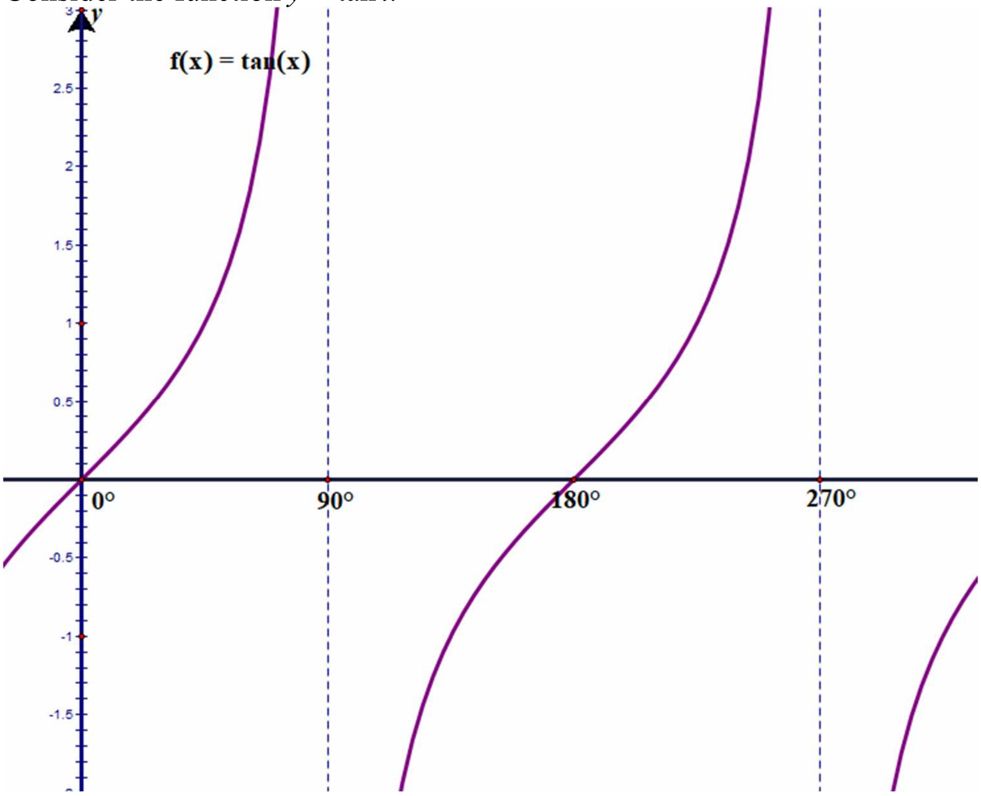
$\lim_{x\to 0^-} \frac{1}{x^2} = \infty$, $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$ (this should be a finite number)

$\because$ $\lim_{x\to 0^-} \frac{1}{x^2} = \lim_{x\to 0^+} \frac{1}{x^2} = \infty$

$\therefore$ $\lim_{x\to 0} \frac{1}{x^2}$ does not exist

Example 7 (3 minutes)

Consider the function $y = \tan x$



$\lim_{x \to 90^{\circ -}} \tan x = \infty$, $\lim_{x \to 90^{\circ +}} \tan x = -\infty$

$\because \lim_{x \to 90^{\circ -}} \tan x \neq \lim_{x \to 90^{\circ +}} \tan x$

$\therefore \lim_{x \to 90^{\circ}} \tan x$ does not exist