Evaluate $\lim_{x\to 0} x\tan\frac{1}{x}$.

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Let $x_n = \dfrac{1}{2n\pi + \frac{\pi}{4}}$, then $x_n \to 0$ as $n \to \infty$

$x_n \tan\dfrac{1}{x_n} = \dfrac{1}{2n\pi + \frac{\pi}{4}} \tan\left(2n\pi + \dfrac{\pi}{4}\right) = \dfrac{1}{2n\pi + \frac{\pi}{4}} \to 0$ as $n \to \infty$

Let $y_n = \dfrac{1}{2n\pi + \tan^{-1}(n)}$, then $y_n \to 0$ as $n \to \infty$

$0 < \tan^{-1}(n) < \dfrac{\pi}{2} \Rightarrow \tan^{-1}(n) < n \Rightarrow 2n\pi + \tan^{-1}(n) < 2n\pi + n \Rightarrow \dfrac{1}{2n\pi + \tan^{-1}(n)} > \dfrac{1}{2n\pi + n}$

$y_n \tan\dfrac{1}{y_n} = \dfrac{1}{2n\pi + \tan^{-1}(n)} \cdot n > \dfrac{n}{2n\pi + n} = \dfrac{1}{2\pi + 1}$

$\therefore \lim_{n\to\infty} y_n \tan\dfrac{1}{y_n} \ge \dfrac{1}{2\pi + 1}$

By Heine's theorem, if $\lim_{x\to 0} x\tan\dfrac{1}{x}$ exists, then $\lim_{n\to\infty} x_n \tan\dfrac{1}{x_n} = \lim_{n\to\infty} y_n \tan\dfrac{1}{y_n}$

However, $\lim_{n\to\infty} x_n \tan\dfrac{1}{x_n} = 0$ and $\lim_{n\to\infty} y_n \tan\dfrac{1}{y_n} \ge \dfrac{1}{2\pi + 1}$

$\therefore \lim_{x\to 0} x\tan\dfrac{1}{x}$ does not exists.