Limit on *e*

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Revision

The limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ (n = 1, 2, 3, $\cdots$) exists and equal to e, where $e \approx 2.71828\cdots$

e is an infinite and non-recurring decimal

$\therefore$ e is an irrational number

We know that $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)=\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}\right)$ (textbook pages 31-32)

Now, there are some new formulae which you must remember:

$e=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y$ (y is a real number)

$e^x=\lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^x=\lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y=\lim_{n\to\infty}\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)=\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{x^r}{r!}\right)$ textbook page 33

Theorem 5.4 on page 32

$\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y$

We simiply this by change variable, let $z=-y$, then as $y\to-\infty$, $z\to\infty$

$$
\begin{aligned}
\lim_{y\to-\infty}\left(1+\frac{1}{y}\right)^y &= \lim_{z\to\infty}\left(1+\frac{1}{-z}\right)^{-z}\\
&=\lim_{z\to\infty}\left(1-\frac{1}{z}\right)^{-z}=\lim_{z\to\infty}\left(\frac{z-1}{z}\right)^{-z}\\
&=\lim_{z\to\infty}\left(\frac{z}{z-1}\right)^{z}\\
&=\lim_{z\to\infty}\left(\frac{(z-1)+1}{z-1}\right)^{(z-1)+1}\\
&=\lim_{z\to\infty}\left(1+\frac{1}{z-1}\right)^{z-1}\cdot\left(1+\frac{1}{z-1}\right)\\
&=\lim_{z-1\to\infty}\left(1+\frac{1}{z-1}\right)^{z-1}\cdot\lim_{z-1\to\infty}\left(1+\frac{1}{z-1}\right)\\
&=e\times 1=e
\end{aligned}
$$

To evaulate $\lim_{y\to0}(1+y)^{\frac{1}{y}}$.

We find the left-hand limit and the right-hand limit

$\lim_{y\to0^+}(1+y)^{\frac{1}{y}}$

By changing variable, $w=\frac{1}{y}$

As $y\to0^+$, $w\to\infty$

$\lim_{y\to0^+}(1+y)^{\frac{1}{y}}=\lim_{w\to\infty}\left(1+\frac{1}{w}\right)^w=e$ (So the right-hand limit is e)

$\lim_{y \to 0^-} (1+y)^{\frac{1}{y}}$, change variable $w = \frac{1}{y}$, as $y \to 0^-$, $w = \frac{1}{y} \to -\infty$

$= \lim_{w \to -\infty} \left(1+\frac{1}{w}\right)^w = e$ (the base of the exponential function)

$\because$ left-hand limit = right-hand limt

$\therefore \lim_{y \to 0} (1+y)^{\frac{1}{y}} = e$

$2 < e \approx 2.71828\cdots < 3$

If $y = e^x \approx (2.71828\cdots)^x$

Then we have the three index laws

(1) $e^a \cdot e^b = e^{a+b}$

(2) $e^a \div e^b = \frac{e^a}{e^b} = e^{a-b}$

(3) If c is a real number then $(e^a)^c = e^{ac}$

(4) $e^0 = 1$

(5) $e^{-a} = \frac{1}{e^a}$

Revision on $y = 10^x$ and $y = \log_{10} x$.

$10^3 = 1000$, then $\log 1000 = 3$

$y = \log_{10} x$ (called the Napoleon logarithms) is the inverse function of $y = 10^x$

Similarly, we define if $x = e^y$, then $y = \log_e x$

This function is called the natural logarithms.

E.g $e^{2.3} \approx 9.97$ (cor. to 3 sig. fig.)

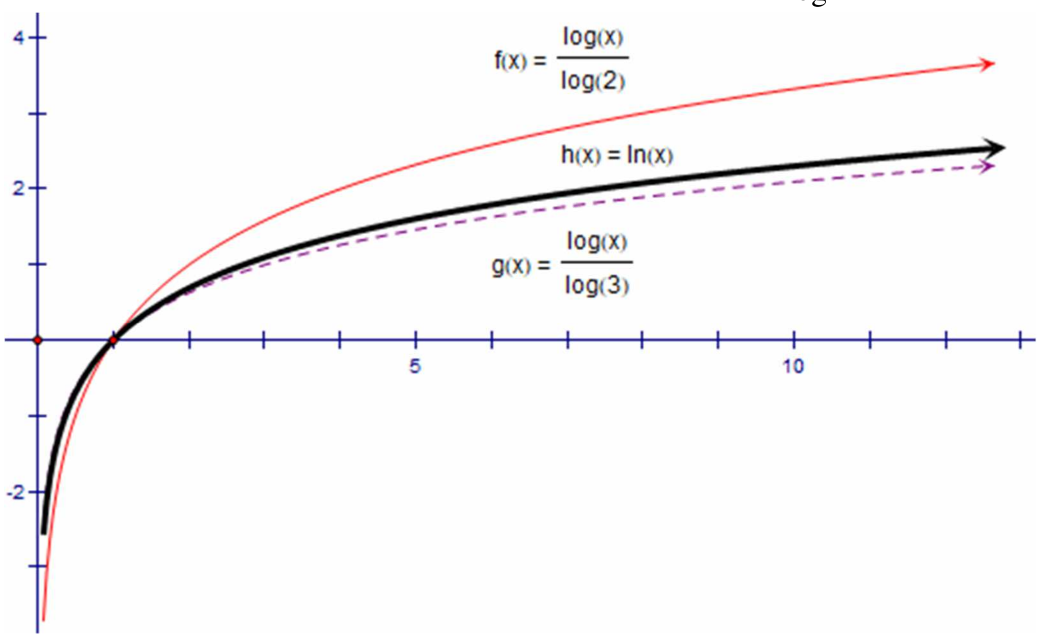
Then $\log_e 9.97 \approx 2.3$ (correct to 1 decimal place)

Usually, we write $\log_e x$ as $y = \ln x = \ell n\, x$

When $x = 0$ or $x < 0$, $y = \ln x$ is undefined.

$y = \ln x$ is defined for $x > 0$ only

Graph of $y = \ln x$, using change of base formula, $y = \log_2 x = \frac{\log x}{\log 2}$, $y = \log_3 x = \frac{\log x}{\log 3}$:



The three laws of $y = \ln x$

If $M, N > 0$

(1) $\ln M + \ln N = \ln(MN)$

(2) $\ln(M - N) = \ln\left(\frac{M}{N}\right)$

(3) If n is any real number, $\ln M^n = n \ln M$

(4) $\ln 1 = 0$

(5) $\ln x = \frac{\log x}{\log e}$ (change of base formula)

Theorem 5.5 on p.5.37

$$\lim_{x\to 0}\frac{e^x - 1}{x} = 1$$

Method 2

Recall the formula $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right)$

This formula will be proved in university.

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x - 1}{x} &= \lim_{x\to 0}\frac{\lim_{n\to\infty}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) - 1}{x} \\
&= \lim_{x\to 0}\frac{\lim_{n\to\infty}\left(\frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right)}{x} \\
&= \lim_{x\to 0}\frac{\lim_{n\to\infty} x\left(\frac{1}{1!} + \frac{x}{2!} + \cdots + \frac{x^{n-1}}{n!}\right)}{x} \\
&= \lim_{x\to 0}\frac{\lim_{n\to\infty}\left(\frac{1}{1!} + \frac{x}{2!} + \cdots + \frac{x^{n-1}}{n!}\right)}{1} \\
&= \lim_{n\to\infty}\left[\lim_{x\to 0}\left(\frac{1}{1!} + \frac{x}{2!} + \cdots + \frac{x^{n-1}}{n!}\right)\right] \\
&= 1
\end{aligned}$$