Differentiation from first principles

Created by Mr. Francis Hung on 20220211. Last updated: 11 February 2022.

Find $\frac{\mathrm{d}x^{-3}}{\mathrm{d}x}$ from first principles.

$$\frac{\mathrm{d}x^{-3}}{\mathrm{d}x}=\lim_{\Delta x\to 0}\frac{\frac{1}{(x+\Delta x)^3}-\frac{1}{x^3}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{x^3-(x+\Delta x)^3}{\Delta x(x+\Delta x)^3x^3}$$

$$=\lim_{\Delta x\to 0}\frac{x^3-\left[x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3\right]}{\Delta x(x+\Delta x)^3x^3}$$

$$=\lim_{\Delta x\to 0}\frac{-3x^2\Delta x-3x(\Delta x)^2-(\Delta x)^3}{\Delta x(x+\Delta x)^3x^3}$$

$$=\lim_{\Delta x\to 0}\frac{-\Delta x\left[3x^2+3x(\Delta x)+(\Delta x)^2\right]}{\Delta x(x+\Delta x)^3x^3}$$

$$=\lim_{\Delta x\to 0}\frac{-\left[3x^2+3x(\Delta x)+(\Delta x)^2\right]}{(x+\Delta x)^3x^3}$$

$$=\frac{-(3x^2+3x\cdot 0+0)}{x^3\cdot x^3}=-\frac{3}{x^4}$$

Find $\frac{\mathrm{d}x^{\frac{1}{3}}}{\mathrm{d}x}$ from first principles.

$$\frac{\mathrm{d}x^{\frac{1}{3}}}{\mathrm{d}x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^{\frac{1}{3}}-x^{\frac{1}{3}}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left[(x+\Delta x)^{\frac{1}{3}}-x^{\frac{1}{3}}\right]}{\Delta x}\times\frac{\left[(x+\Delta x)^{\frac{2}{3}}+(x+\Delta x)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right]}{\left[(x+\Delta x)^{\frac{2}{3}}+(x+\Delta x)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right]}$$

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)-x}{\Delta x\left[(x+\Delta x)^{\frac{2}{3}}+(x+\Delta x)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right]}$$, by using the identity $(a-b)(a^2+ab+b^2)=a^3-b^3$

$$=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x\left[(x+\Delta x)^{\frac{2}{3}}+(x+\Delta x)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right]}$$

$$=\lim_{\Delta x\to 0}\frac{1}{(x+\Delta x)^{\frac{2}{3}}+(x+\Delta x)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}}$$

$$=\frac{1}{x^{\frac{2}{3}}+x^{\frac{2}{3}}+x^{\frac{2}{3}}}=\frac{1}{3}x^{-\frac{2}{3}}$$

Exercise

Find $\frac{\mathrm{d}x^{-5}}{\mathrm{d}x}$ and $\frac{\mathrm{d}x^{\frac{2}{5}}}{\mathrm{d}x}$ from first principles.

Answers: $-5x^{-6}$; $\frac{2}{5}x^{-\frac{3}{5}}$.