Example on differentiability

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Let $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

(a) Find f'(x) for $x \neq 0$.

(b) Find f'(0) .

(c) Show that f'(x) is not continuous at $x = 0$.

(a) For $x \neq 0$, $f'(x) = 2x\sin\dfrac{1}{x} + x^2 \cdot \left(\cos\dfrac{1}{x}\right) \cdot \left(-\dfrac{1}{x^2}\right) = 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x}$

(b) $f'(0) = \lim\limits_{h\to 0} \dfrac{h^2 \sin\frac{1}{h} - 0}{h} = \lim\limits_{h\to 0} h\sin\dfrac{1}{h}$

$\because -1 \le \sin\dfrac{1}{h} \le 1$ and $\lim\limits_{h\to 0} h = 0$

$\therefore \lim\limits_{h\to 0} h\sin\dfrac{1}{h} = 0$

$\Rightarrow f'(0) = 0$

(c) $\lim\limits_{x\to 0} f'(x) = \lim\limits_{h\to 0}\left(2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x}\right)$

$= \lim\limits_{h\to 0}\left(0 - \cos\dfrac{1}{x}\right)$

$= -\lim\limits_{h\to 0} \cos\dfrac{1}{x}$

Let $h = \dfrac{1}{2n\pi}$, where n is a non-zero integer.

Then $h \to 0^+$ if and only if $n \to \infty$

$\lim\limits_{h\to 0+} \cos\dfrac{1}{h} = \lim\limits_{n\to\infty} \cos 2n\pi = 1$

Let $h = \dfrac{1}{2n\pi + \pi}$, where n is a non-zero integer.

Then $h \to 0^+$ if and only if $n \to \infty$

$\lim\limits_{h\to 0+} \cos\dfrac{1}{h} = \lim\limits_{n\to\infty} \cos(2n+1)\pi = -1$

If $\lim\limits_{h\to 0} \cos\dfrac{1}{h}$ exists, then $\lim\limits_{h\to 0+} \cos\dfrac{1}{h}$ exists and it must be unique.

However, $\lim\limits_{n\to\infty} \cos 2n\pi$ and $\lim\limits_{n\to\infty} \cos(2n+1)\pi$ tends to two different limits, so $\lim\limits_{h\to 0} \cos\dfrac{1}{h}$ does not exist.

$\therefore \lim\limits_{x\to 0} f'(x)$ does not exist.

f'(x) is not continuous at $x = 0$.