L' Hôpital's Rule

Created by Francis Hung on 26 June 2008 Last updated: February 11, 2022

Let f and g be differentiable on $(a - \delta, a + \delta) \setminus \{a\}$ with $g'(x) \neq 0$ on this set. Then

(i) If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ and if $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \ell$ exists,

then $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and equals to ℓ

(ii) If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$ and if $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \ell$ exists,

then $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and equals to ℓ.

Proof: (i) Take x s.t. $a < x < a + \delta$

Now f and g continuous on $[a, x]$ and differentiable on (a, x)

$[f(x)$ and $g(x) \to 0$ as $x \to a]$

By Cauchy's mean value theorem on $[a, x]$

$\exists c_x \in (a, x)$ s.t. $[f(x) - f(a)]g'(c_x) = [g(x) - g(a)]f'(c_x)$

Note: $g(x) \neq 0$; otherwise $\exists \alpha \in (a, x)$ s.t. $g'(\alpha) = 0$ contradiction

$\therefore$ we get $\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)} \to \ell$ as $x \to a+$ $\quad (c_x \to a)$

Similarly $\lim_{x \to a-} \frac{f(x)}{g(x)} = \ell$

$\therefore \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \ell$

(ii) Given $\varepsilon > 0$, first choose $x_2 \in (a, a + \delta)$ s.t. whenever $a < x \leq x_2$,

We have $\left| \frac{f'(x)}{g'(x)} - \ell \right| < \varepsilon$ and $g(x) > 0$

Now choose $x_1 \in (a, x_2)$ s.t. $g(x) > \max \left\{ \frac{g(x_2)}{\varepsilon}, \frac{f(x_2)}{\varepsilon} \right\}$ whenever $a < x \leq x_1$

Apply Cauchy's mean value theorem on $[x, x_2]$

$\exists\, c_x \in (x, x_2)$ s.t. $[f(x_2) - f(x)]g'(c_x) = [g(x_2) - g(x)]f'(c_x)$

$$\frac{f(x)}{g(x)} = \frac{f(x_2)}{g(x)} + \frac{f'(c_x)}{g'(c_x)} - \frac{g(x_2)f'(c_x)}{g(x)g'(c_x)}$$

$$\left| \frac{f(x)}{g(x)} - \ell \right| \leq \left| \frac{f(x_2)}{g(x)} \right| + \left| \frac{f'(c_x)}{g'(c_x)} - \ell \right| + \left| \frac{g(x_2)f'(c_x)}{g(x)g'(c_x)} \right|$$

$\leq \varepsilon + \varepsilon + \varepsilon(|\ell| + \varepsilon) < (|\ell| + 3)\,\varepsilon$ assuming $\varepsilon < 1$

$\therefore \frac{f(x)}{g(x)} \to \ell$ as $x \to a^+$

Similarly $\lim_{x \to a-} \frac{f(x)}{g(x)} = \ell$

(I) Rules on infinity (∞)

∞ is not a number, for any real number x, $-\infty < x < \infty$

(a) $\infty + \infty = \infty$

(b) $\infty \times \infty = \infty$

(c) $\frac{1}{\infty} = 0$

(d) $\lim_{x\to 0+} \frac{1}{x} = \infty$; $\lim_{x\to 0-} \frac{1}{x} = -\infty$

(e) $\infty^\infty = \infty$

(f) $a^\infty = \infty$ for $a > 1$

(g) $a^\infty = 0$ for $0 < a < 1$

(h) $0^a = 0$ for $a > 0$, a can be ∞

(i) $1^a = 1$ for any number a, $a \neq \infty$

(II) Indeterminate Forms

(a) $\frac{0}{0}$

(b) $\frac{\infty}{\infty}$

(c) $0 \times \infty$

(d) $\infty - \infty$

(e) 0^0

(f) ∞^0

(g) 1^∞

(III) If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$; (a can be any number or ∞ or $-\infty$)

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

(IV) Examples and exercises:

(1) $\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$ (type $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{-x\sin x + \cos x - \cos x}{3x^2}$ (L' Hôpital's rule)

$= \lim_{x\to 0} \frac{-\sin x}{3x}$ (type $\frac{0}{0}$)

$= \lim_{x\to 0} \frac{-\cos x}{3}$ (L' Hôpital's rule)

$= -\frac{1}{3}$

Exercise 1.1 $\lim_{x\to 0}\frac{\cos\left[\left(\frac{\pi}{2}\right)\cos x\right]}{\sin^2 x}$

Exercise 1.2 $\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x}$

Exercise 1.3 If $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$, find a and b.

(2) $\lim_{x\to a+}\frac{\log(x-a)}{\log(e^x-e^a)}$ (type $\frac{\infty}{\infty}$)

$=\lim_{x\to a+}\frac{\frac{1}{x-a}}{\frac{e^x}{e^x-e^a}}$ (L' Hôpital's rule)

$=\lim_{x\to a+}\frac{e^x-e^a}{e^x(x-a)}$ (type $\frac{0}{0}$)

$=\lim_{x\to a+}\frac{e^x}{e^x(x-a)+e^x}$ (L' Hôpital's rule)

$=\frac{e^a}{e^a}=1$

Exercise 2.1 $\lim_{x\to \infty}\frac{\log(1+e^x)}{x}$

Exercise 2.2 $\lim_{x\to \frac{\pi}{2}}\frac{\log\tan^2 x}{\log\left(x-\frac{\pi}{2}\right)^2}$

(3) $\lim_{x\to 0+}x\log x$ (type $0\times-\infty$)

$=\lim_{x\to 0+}\frac{\log x}{\frac{1}{x}}$ (type $\frac{\infty}{\infty}$)

$=\lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$ (L' Hôpital's rule)

$=\lim_{x\to 0+}(-x)=0$

Exercise 3.1 $\lim_{x\to a}(a-x)\tan\left(\frac{\pi x}{2a}\right)$

Exercise 3.2 $\lim_{x\to 0}\sin(\sin x)\tan\left(x-\frac{\pi}{2}\right)$

(4) $\lim_{x\to 2}\left[\frac{1}{x-2}-\frac{1}{\log(x-1)}\right]$ (type $\infty-\infty$)

$=\lim_{x\to 2}\frac{\log(x-1)-(x-2)}{(x-2)\log(x-1)}$ (type $\frac{0}{0}$)

$=\lim_{x\to 2}\frac{\frac{1}{x-1}-1}{\frac{x-2}{x-1}+\log(x-1)}$ (L' Hôpital's rule)

$=\lim_{x\to 2}\frac{2-x}{x-2+(x-1)\log(x-1)}$ (type $\frac{0}{0}$)

$= \lim_{x \to 2} \frac{-1}{1+1+\log(x-1)}$ (L' Hôpital's rule)

$= -\frac{1}{2}$

Exercise 4.1 $\lim_{x \to 0} \left(\frac{a}{x} - \cot \frac{x}{a} \right)$

Exercise 4.2 $\lim_{x \to \frac{\pi}{2}} \left(\frac{2x}{\pi} \sec x - \tan x \right)$

Exercise 4.3 $\lim_{x \to 0} \left(\frac{x-1}{2x^2} + \frac{e^{-x}}{2x \sinh x} \right)$, where $\sinh z = \frac{1}{2}(e^z - e^{-z})$.

(5) Evaluate $\lim_{x \to 0} (\cos x)^{1/x^2}$.

Let $y = (\cos x)^{1/x^2}$

$\log y = \frac{\log(\cos x)}{x^2}$

$\lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log \cos x}{x^2}$ (type $\frac{0}{0}$)

$= \lim_{x \to 0} \frac{-\tan x}{2x}$ (L' Hôpital's rule), (type $\frac{0}{0}$)

$= \lim_{x \to 0} \left(-\frac{\sec^2 x}{2} \right)$ (L' Hôpital's rule)

$\log \lim_{x \to 0} y = -\frac{1}{2}$

$\lim_{x \to 0} y = e^{-\frac{1}{2}}$

Exercise 5.1 $\lim_{x \to 0} (1 + \sin x)^{\cot x}$ (type 1^∞)

Exercise 5.2 $\lim_{x \to \infty} x^{1/x}$ (type ∞^0)

Exercise 5.3 $\lim_{x \to a+} (x-a)^{x-a}$ (type 0^0)

Exercise 1.1 $\lim_{x\to 0}\frac{\cos\left[\left(\frac{\pi}{2}\right)\cos x\right]}{\sin^2 x}$ (type $\frac{0}{0}$)

$=\lim_{x\to 0}\frac{-\sin\left[\left(\frac{\pi}{2}\right)\cos x\right]\frac{\pi}{2}\left(-\sin x\right)}{2\sin x\cos x}$ (L' Hôpital's rule)

$=\frac{\pi}{4}\lim_{x\to 0}\frac{\sin\left[\left(\frac{\pi}{2}\right)\cos x\right]}{\cos x}$ (L' Hôpital's rule)

$=\frac{\pi}{4}\cdot\frac{\sin\frac{\pi}{2}}{\cos 0}=\frac{\pi}{4}$

Exercise 1.2 $\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x}$

Note that $\lim_{x\to 0+}(1+x)^{1/x}=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y$

$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

and $\lim_{x\to 0-}(1+x)^{1/x}=\lim_{y\to -\infty}\left(1+\frac{1}{y}\right)^y$

$=\lim_{z\to\infty}\left(1-\frac{1}{z}\right)^{-z}$, where $y=-z$

$=\lim_{z\to\infty}\frac{1}{\left(\frac{z-1}{z}\right)^z}$

$=\lim_{z\to\infty}\left(\frac{z}{z-1}\right)^z$

$=\lim_{z\to\infty}\left(1+\frac{1}{z-1}\right)^{z-1}\left(1+\frac{1}{z-1}\right)$

$=\lim_{z\to\infty}\left(1+\frac{1}{z-1}\right)^{z-1}\cdot 1$

$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, where $n=z-1$

$=e$

$\therefore\ \lim_{x\to 0+}(1+x)^{1/x}=\lim_{x\to 0-}(1+x)^{1/x}=e$

Now let $y=(1+x)^{1/x}$

$\ln y=\frac{\ln(1+x)}{x}$

Differentiate w.r.t. x

$\frac{y'}{y}=\frac{\frac{x}{1+x}-\ln(1+x)}{x^2}$

$y'=\frac{x-(1+x)\ln(1+x)}{x^2(1+x)}\cdot(1+x)^{1/x}$ …… (*)

$\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x}$ (type $\frac{e-e}{0}=\frac{0}{0}$)

$=\lim_{x\to 0}-\frac{d}{dx}(1+x)^{1/x}$ (L' Hôpital's rule)

$= -\lim_{x\to 0} \frac{x-(1+x)\ln(1+x)}{x^2(1+x)} \cdot (1+x)^{1/x}$ (by (*))

$= -e\lim_{x\to 0} \frac{x-(1+x)\ln(1+x)}{x^2(1+x)}$ (type $\frac{0}{0}$)

$= -e\lim_{x\to 0} \frac{1-1-\ln(1+x)}{2x+3x^2}$ (L' Hôpital's rule)

$= e\lim_{x\to 0} \frac{\ln(1+x)}{2x+3x^2}$ (type $\frac{0}{0}$)

$= e\lim_{x\to 0} \frac{\frac{1}{1+x}}{2+6x}$ (L' Hôpital's rule)

$= \frac{e}{2}$

Exercise 1.3 If $\lim_{x\to 0} \frac{x(1+a\cos x)-b\sin x}{x^3} = 1$, find a and b. (type $\frac{0}{0}$)

$\lim_{x\to 0} \frac{(1+a\cos x)-ax\sin x-b\cos x}{3x^2} = 1$, (L' Hôpital's rule)

$\lim_{x\to 0} \frac{1+(a-b)\cos x-ax\sin x}{3x^2} = 1$

In order that the limit exist, and equal to 1, it must be of the type $\frac{0}{0}$

$\therefore$ $1 + a - b = 0$ ……… (1)

$\lim_{x\to 0} \frac{1-\cos x-ax\sin x}{3x^2} = 1$, (type $\frac{0}{0}$)

$\lim_{x\to 0} \frac{\sin x-a\sin x-ax\cos x}{6x} = 1$, (L' Hôpital's rule)

$\lim_{x\to 0} \frac{(1-a)\sin x-ax\cos x}{6x} = 1$, (type $\frac{0}{0}$)

$\lim_{x\to 0} \frac{(1-a)\cos x-a\cos x+ax\sin x}{6} = 1$, (L' Hôpital's rule)

$\lim_{x\to 0} \frac{(1-2a)\cos x+ax\sin x}{6} = 1$

$\frac{(1-2a)}{6} = 1$ ………(2)

$a = -\frac{5}{2}$

Sub. into (1), $b = 1 + a = -\frac{3}{2}$

Exercise 2.1 $\lim_{x\to\infty} \frac{\log(1+e^x)}{x}$ (type $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \frac{\frac{1}{1+e^x}\cdot e^x}{1}$ (L' Hôpital's rule)

$= \lim_{x\to\infty} \frac{e^x}{1+e^x}$ (type $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \frac{e^x}{e^x} = 1$ (L' Hôpital's rule)

Exercise 2.2 $\lim_{x\to\frac{\pi}{2}} \frac{\log\tan^2 x}{\log\left(x-\frac{\pi}{2}\right)^2}$ (type $\frac{\infty}{\infty}$)

$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{2\tan x\sec^2 x}{\tan^2 x}}{\frac{2\left(x-\frac{\pi}{2}\right)}{\left(x-\frac{\pi}{2}\right)^2}}$ (L' Hôpital's rule)

$= \lim_{x\to\frac{\pi}{2}} \frac{\left(x-\frac{\pi}{2}\right)}{\sin x\cos x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{1}{\sin x} \cdot \lim_{x\to\frac{\pi}{2}} \frac{x-\frac{\pi}{2}}{\cos x}$ (type $\frac{0}{0}$)

$= \lim_{x\to\frac{\pi}{2}} \frac{1}{-\sin x} = -1$ (L' Hôpital's rule)

Exercise 3.1 $\lim_{x\to a}(a-x)\tan\left(\frac{\pi x}{2a}\right)$ (type $0\times\infty$)

$= \lim_{x\to a} \frac{\tan\left(\frac{\pi x}{2a}\right)}{\frac{1}{a-x}}$ (type $\frac{\infty}{\infty}$)

$= \lim_{x\to a} \frac{\frac{\pi}{2a}\sec^2\left(\frac{\pi x}{2a}\right)}{\frac{1}{(a-x)^2}}$

$= \lim_{x\to a} \frac{\pi}{2a} \cdot \frac{(a-x)^2}{\cos^2\left(\frac{\pi x}{2a}\right)}$ (type $\frac{0}{0}$)

$= \lim_{x\to a} \frac{\pi}{2a} \cdot \frac{-2(a-x)}{-2\cos\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi x}{2a}\right)\cdot\frac{\pi}{2a}}$ (L' Hôpital's rule)

$= \lim_{x\to a} \frac{2(a-x)}{2\cos\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi x}{2a}\right)}$

$= 2\lim_{x\to a} \frac{a-x}{\sin\left(\frac{\pi x}{a}\right)}$ (type $\frac{0}{0}$)

$= 2\lim_{x\to a} \frac{-1}{\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)}$ (L' Hôpital's rule)

$= \frac{2a}{\pi}$

Exercise 3.2 $\lim_{x\to 0}\sin(\sin x)\tan\left(x-\frac{\pi}{2}\right)$ (type $0\times-\infty$)

$= \lim_{x\to 0}\sin(\sin x)\cdot(-\cot x)$

$= -\lim_{x\to 0} \frac{\sin(\sin x)}{\tan x}$ (type $\frac{0}{0}$)

$= -\lim_{x\to 0} \frac{\cos(\sin x)\cos x}{\sec^2 x}$ (L' Hôpital's rule)

$= -\lim_{x\to 0}\cos(\sin x)\cos^3 x = -1$

Exercise 4.1 $\lim_{x\to 0}\left(\frac{a}{x}-\cot\frac{x}{a}\right)$

$= \lim_{x\to 0} \frac{a\tan\frac{x}{a}-x}{x\tan\frac{x}{a}}$ (type $\frac{0}{0}$)

$=\lim_{x\to 0}\frac{a\sec^2\frac{x}{a}\cdot\frac{1}{a}-1}{\tan\frac{x}{a}+x\sec^2\frac{x}{a}\cdot\frac{1}{a}}$ (L' Hôpital's rule)

$=\lim_{x\to 0}\frac{\tan^2\frac{x}{a}}{\tan\frac{x}{a}+\frac{x}{a}\sec^2\frac{x}{a}}$ ($\because \sec^2\frac{x}{a}-1=\tan^2\frac{x}{a}$)

$=\lim_{x\to 0}\frac{\sin^2\frac{x}{a}}{\sin\frac{x}{a}\cos\frac{x}{a}+\frac{x}{a}}$

$=\lim_{x\to 0}\frac{\frac{1}{2}\left(1-\cos\frac{2x}{a}\right)}{\frac{1}{2}\sin\frac{2x}{a}+\frac{x}{a}}$

$=\lim_{x\to 0}\frac{1-\cos\frac{2x}{a}}{\sin\frac{2x}{a}+\frac{2x}{a}}$ (type $\frac{0}{0}$)

$=\lim_{x\to 0}\frac{\frac{2}{a}\sin\frac{2x}{a}}{\frac{2}{a}\cos\frac{2x}{a}+\frac{2}{a}}=0$ (L' Hôpital's rule)

Exercise 4.2 $\lim_{x\to\frac{\pi}{2}}\left(\frac{2x}{\pi}\sec x-\tan x\right)$

$=\lim_{x\to\frac{\pi}{2}}\frac{2x-\pi\sin x}{\pi\cos x}$ (type $\frac{0}{0}$)

$=\lim_{x\to\frac{\pi}{2}}\frac{2-\pi\cos x}{-\pi\sin x}$ (L' Hôpital's rule)

$=-\frac{2}{\pi}$

Exercise 4.3 $\lim_{x\to 0}\left(\frac{x-1}{2x^2}+\frac{e^{-x}}{2x\sinh x}\right)$, where $\sinh z=\frac{1}{2}(e^z-e^{-z})$.

$=\lim_{x\to 0}\frac{(x-1)\sinh x+xe^{-x}}{2x^2\sinh x}$

$=\lim_{x\to 0}\frac{(x-1)\cdot\frac{1}{2}\left(e^x-e^{-x}\right)+xe^{-x}}{2x^2\cdot\frac{1}{2}\left(e^x-e^{-x}\right)}$

$=\lim_{x\to 0}\frac{(x-1)\left(e^x-e^{-x}\right)+2xe^{-x}}{2x^2\left(e^x-e^{-x}\right)}$

$=\lim_{x\to 0}\frac{(x-1)e^x+(x+1)e^{-x}}{2x^2\left(e^x-e^{-x}\right)}$

$=\lim_{x\to 0}\frac{(x-1)e^{2x}+x+1}{2x^2\left(e^{2x}-1\right)}$ (type $\frac{0}{0}$)

$=\frac{1}{2}\lim_{x\to 0}\frac{e^{2x}+2(x-1)e^{2x}+1}{2x\left(e^{2x}-1\right)+x^2\left(2e^{2x}\right)}$ (L' Hôpital's rule)

$=\frac{1}{4}\lim_{x\to 0}\frac{(2x-1)e^{2x}+1}{\left(x^2+x\right)e^{2x}-x}$ (type $\frac{0}{0}$)

$=\frac{1}{4}\lim_{x\to 0}\frac{2e^{2x}+(2x-1)\cdot 2\cdot e^{2x}}{(2x+1)e^{2x}+2\left(x^2+x\right)e^{2x}-1}$ (L' Hôpital's rule)

$=\frac{1}{4}\lim_{x\to 0}\frac{4xe^{2x}}{\left(2x^2+4x+1\right)e^{2x}-1}$

$=\lim_{x\to 0}\frac{x}{2x^2+4x+1-e^{-2x}}$ (type $\frac{0}{0}$)

$$= \lim_{x \to 0} \frac{1}{4x + 4 + 2e^{-2x}}$$ (L' Hôpital's rule)

$$= \frac{1}{6}$$

Exercise 5.1 $\lim_{x \to 0} (1 + \sin x)^{\cot x}$ (type 1^∞)

Let $y = (1 + \sin x)^{\cot x}$

$\log y = \cot x \log(1 + \sin x)$

$$\log y = \frac{\log(1 + \sin x)}{\tan x}$$

$$\lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log(1 + \sin x)}{\tan x}$$ (type $\frac{0}{0}$)

$$= \lim_{x \to 0} \frac{\frac{\cos x}{1 + \sin x}}{\sec^2 x}$$ (L' Hôpital's rule)

$$= \lim_{x \to 0} \frac{\cos^3 x}{1 + \sin x} = 1$$

$\log \lim_{x \to 0} y = 1 \Rightarrow \lim_{x \to 0} y = e$

Exercise 5.2 $\lim_{x \to \infty} x^{1/x}$ (type ∞^0)

Let $y = x^{1/x}$

$$\log y = \frac{\log x}{x}$$

$$\lim_{x \to \infty} \log y = \lim_{x \to \infty} \frac{\log x}{x}$$ (type $\frac{\infty}{\infty}$)

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$ (L' Hôpital's rule)

$\log \lim_{x \to 0} y = 0 \Rightarrow \lim_{x \to 0} y = e^0 = 1$

Exercise 5.3 $\lim_{x \to a+} (x - a)^{x-a}$ (type 0^0)

Let $y = (x - a)^{x - a}$

$\log y = (x - a)\log(x - a)$

$$\lim_{x \to a+} \log y = \lim_{x \to a+} \frac{\log(x - a)}{\frac{1}{x-a}}$$ (type $\frac{\infty}{\infty}$)

$$\log \lim_{x \to a+} y = \lim_{x \to a+} \frac{\frac{1}{x-a}}{-\frac{1}{(x-a)^2}}$$ (L' Hôpital's rule)

$$= -\lim_{x \to a+} (x - a) = 0$$

$\lim_{x \to a+} y = e^0 = 1$