Differentiation formulae

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Let c be a constant, f(x), g(x) and u be differentiable functions of x and $g(x) \neq 0$.

1. $\frac{\mathrm{d}c}{\mathrm{d}x} = 0$

2. $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)+\mathrm{g}(x)\right] = \frac{\mathrm{df}(x)}{\mathrm{d}x} + \frac{\mathrm{dg}(x)}{\mathrm{d}x}$

3. $\frac{\mathrm{d}}{\mathrm{d}x}\left[c\mathrm{f}(x)\right] = c\frac{\mathrm{df}(x)}{\mathrm{d}x}$.

4. $\frac{\mathrm{d}x}{\mathrm{d}x} = 1$

5. If n is a positive integer, then $\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^n - 1$. e.g. $\frac{\mathrm{d}x^5}{\mathrm{d}x} = 5x^4$.

   This can be proved by binomial theorem from first principles.

   If n is a negative integer, then $\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^n - 1$. e.g. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^3}\right) = \frac{\mathrm{d}x^{-3}}{\mathrm{d}x} = -3x^{-4} = -\frac{3}{x^4}$.

   If n is a rational number, i.e. $n = \frac{p}{q}$, where p and q are relatively prime integer,

   then $\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^n - 1$. e.g. $\frac{\mathrm{d}\sqrt{x}}{\mathrm{d}x} = \frac{\mathrm{d}x^{\frac{1}{2}}}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

   If n is a real number, then $\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^n - 1$. e.g. $\frac{\mathrm{d}x^\pi}{\mathrm{d}x} = \pi x^{\pi-1}$.

   $\frac{\mathrm{d}u^n}{\mathrm{d}x} = nu^{n-1}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$. eg. $\frac{\mathrm{d}(2x+3)^4}{\mathrm{d}x} = 4(2x+3)^3\cdot\frac{\mathrm{d}(2x+3)}{\mathrm{d}x} = 8(2x+3)^3$

6. $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\times\mathrm{g}(x)\right] = \mathrm{f}(x)\frac{\mathrm{dg}(x)}{\mathrm{d}x} + \mathrm{g}(x)\frac{\mathrm{df}(x)}{\mathrm{d}x}$

7. $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right] = \frac{\mathrm{g}(x)\frac{\mathrm{df}(x)}{\mathrm{d}x} - \mathrm{f}(x)\frac{\mathrm{dg}(x)}{\mathrm{d}x}}{\left[\mathrm{g}(x)\right]^2}$

8. **<u>Chain rule</u>**: $\frac{\mathrm{df}(u)}{\mathrm{d}x} = \frac{\mathrm{df}(u)}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$. e.g. $\frac{\mathrm{d}\sqrt{2x-3}}{\mathrm{d}x} = \frac{1}{2}(2x-3)^{-\frac{1}{2}}\cdot\frac{\mathrm{d}(2x-3)}{\mathrm{d}x} = \frac{1}{\sqrt{2x-3}}$.

9. Differentiation of **inverse function**: $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$. e.g. $x = y + y^3$, $\frac{\mathrm{d}x}{\mathrm{d}y} = 1 + 3y^2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+3y^2}$.

10. Differentiation of **implicit function**: $\frac{\mathrm{d}\left(x^m y^n\right)}{\mathrm{d}x} = mx^{m-1}y^n + nx^m y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}$.

    e.g. $x^2y^3 + y = x$, differentiate both sides w.r.t. x .

    $$\frac{\mathrm{d}\left(x^2y^3\right)}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}x}$$

    $$2xy^3 + 3x^2y^2\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

    $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-2xy^3}{3x^2y^2+1}$$

11. $\frac{\mathrm{d}\sin x}{\mathrm{d}x} = \cos x$, $\frac{\mathrm{d}\sin u}{\mathrm{d}x} = \cos u \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

$\frac{\mathrm{d}\cos x}{\mathrm{d}x} = -\sin x$, $\frac{\mathrm{d}\cos u}{\mathrm{d}x} = -\sin u \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

$\frac{\mathrm{d}\tan x}{\mathrm{d}x} = \sec^2 x$, $\frac{\mathrm{d}\tan u}{\mathrm{d}x} = \sec^2 u \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

$\frac{\mathrm{d}\sec x}{\mathrm{d}x} = \sec x \tan x$, $\frac{\mathrm{d}\sec u}{\mathrm{d}x} = \sec u \tan u \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

$\frac{\mathrm{d}\csc x}{\mathrm{d}x} = -\csc x \cot x$, $\frac{\mathrm{d}\csc u}{\mathrm{d}x} = -\csc u \cot u \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

$\frac{\mathrm{d}\cot x}{\mathrm{d}x} = -\csc^2 x$, $\frac{\mathrm{d}\cot u}{\mathrm{d}x} = -\csc^2 u \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

12. $\frac{\mathrm{d}(\arcsin x)}{\mathrm{d}x} = \frac{\mathrm{d}(\sin^{-1} x)}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$. It can be proved by **9. Inverse function**.

$\frac{\mathrm{d}(\text{arc}\cos x)}{\mathrm{d}x} = \frac{\mathrm{d}(\cos^{-1} x)}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

$\frac{\mathrm{d}(\text{arc}\tan x)}{\mathrm{d}x} = \frac{\mathrm{d}(\tan^{-1} x)}{\mathrm{d}x} = \frac{1}{1+x^2}$.

Proof: $y = \text{arc}\tan x$

$\tan y = x$, differentiate both sides w.r.t. x.

$\sec^2 y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 1$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

$\frac{\mathrm{d}(\text{arc}\sec x)}{\mathrm{d}x} = \frac{\mathrm{d}(\sec^{-1} x)}{\mathrm{d}x} = \frac{1}{x\sqrt{x^2-1}}$ for $x < -1$ or $x > 1$.

$\frac{\mathrm{d}(\text{arc}\csc x)}{\mathrm{d}x} = \frac{\mathrm{d}(\csc^{-1} x)}{\mathrm{d}x} = -\frac{1}{x\sqrt{x^2-1}}$ for $x < -1$ or $x > 1$.

$\frac{\mathrm{d}(\text{arc}\cot x)}{\mathrm{d}x} = \frac{\mathrm{d}(\cot^{-1} x)}{\mathrm{d}x} = -\frac{1}{1+x^2}$

13. $\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$. $\frac{\mathrm{d}e^u}{\mathrm{d}x} = e^u \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$. e.g. $\frac{\mathrm{d}e^{2x}}{\mathrm{d}x} = 2e^{2x}$

$\frac{\mathrm{d}\ln x}{\mathrm{d}x} = \frac{1}{x}$. $\frac{\mathrm{d}\ln u}{\mathrm{d}x} = \frac{1}{u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$. e.g. $\frac{\mathrm{d}\ln\sqrt{3x-1}}{\mathrm{d}x} = \frac{1}{2} \cdot \frac{\mathrm{d}\ln(3x-1)}{\mathrm{d}x} = \frac{1}{2(3x-1)}$

14. If a is a positive constant and $a \neq 1$, then $\frac{\mathrm{d}a^x}{\mathrm{d}x} = (\ln a) \cdot a^x$. $\frac{\mathrm{d}a^u}{\mathrm{d}x} = (\ln a) \cdot a^u \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

15. If a is a positive constant and $a \neq 1$, then $\frac{\mathrm{d}\log_a x}{\mathrm{d}x} = \frac{1}{x\ln a}$. $\frac{\mathrm{d}\log_a u}{\mathrm{d}x} = \frac{1}{u\ln a} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

16. Differentiate $y = x^{\frac{1}{x}} \Rightarrow \ln y = \frac{\ln x}{x}$

$$\frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2} \Rightarrow \frac{\mathrm{d}x^{\frac{1}{x}}}{\mathrm{d}x} = \frac{1-\ln x}{x^2} \cdot x^{\frac{1}{x}}$$