Differentiation of composite functions

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Let f, g, h be continuous real functions such that $\forall x \in \mathbf{R}$: $h(x) = f \circ g(x) = f(g(x))$

Then $h'(x) = f'(g(x))g'(x)$, where h' denotes the derivative of h .

This is often refer to as the chain rule for differentiation.

Proof: Let $g(x) = y$, and let $g(x + \Delta x) = y + \Delta y$

$\Rightarrow \Delta y = g(x + \Delta x) - g(x)$

Thus: $\Delta y \to 0$ as $\Delta x \to 0$

and $\frac{\Delta y}{\Delta x} \to \frac{dy}{dx} = g'(x)$ …… (1)

There are two cases to consider:

Case 1

Suppose $g'(x) \neq 0$ and that Δx is small but non-zero.

Thus $\Delta y \neq 0$ from (1) above, and :

$$\lim_{\Delta x \to 0} \frac{h(x+\Delta x)-h(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x))-f(g(x))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x))-f(g(x))}{g(x+\Delta x)-g(x)} \cdot \frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(y+\Delta y)-f(y)}{\Delta y} \cdot \frac{\Delta y}{\Delta x}$$

$$= f'(y)g'(x)$$

Hence the result.

Case 2

Now suppose $g'(x) = 0$ and that Δx is small but non-zero.

Again, there are two possibilities:

Case 2a

If $\Delta y = 0$, then $\frac{h(x+\Delta x)-h(x)}{\Delta x} = 0$

Hence the result.

Case 2b

If $\Delta y \neq 0$, then $\frac{h(x+\Delta x)-h(x)}{\Delta x} = \frac{f(y+\Delta y)-f(y)}{\Delta y} \cdot \frac{\Delta y}{\Delta x}$

As $\Delta y \to 0$:

(1): $\frac{f(y+\Delta y)-f(y)}{\Delta y} \to f'(y)$

(2): $\frac{\Delta y}{\Delta x} \to 0$

Thus: $\lim_{\Delta x \to 0} \frac{h(x+\Delta x)-h(x)}{\Delta x} = f'(y)\, g'(x) = 0$

Again, hence the result.

All cases have been covered, the proof is complete.