Differentiation of inverse trigonometric functions

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We have learnt the differentiation of inverse function.

Example Given $x = y^3 + 2y$. Find $\frac{dx}{dy}$. Hence find $\frac{dy}{dx}$.

$$\frac{dx}{dy} = 3y^2 + 2$$

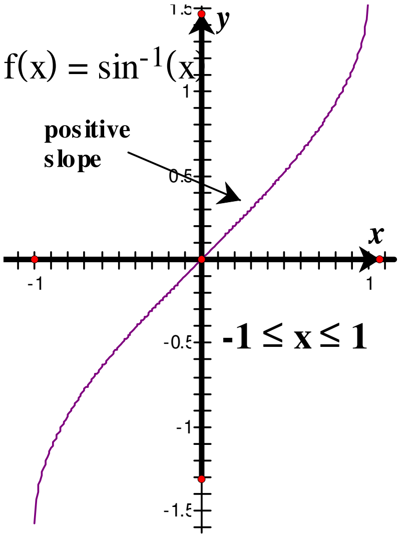
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2 + 2}$$

Now, given $y = \sin^{-1} x$ = arc sin (x). Find $\frac{dy}{dx}$. (Note that $\sin^{-1} x \neq \frac{1}{\sin x}$)

$x = \sin y$

$$\frac{dx}{dy} = \cos y \cdot \frac{dy}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y}$$



Can you express $\frac{dy}{dx}$ in terms of x only?

$\because x = \sin y$

$\therefore \cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \text{ or } -\frac{1}{\sqrt{1-x^2}} \text{ (rejected)}$$

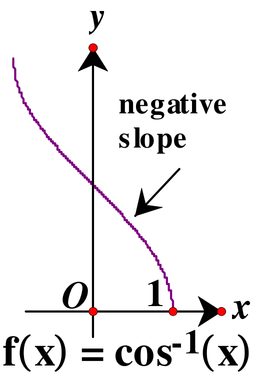
If u is a function of x, then $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

Given $y = \cos^{-1} x$ = arc cos (x). Find $\frac{dy}{dx}$.

$x = \cos y$

$$\frac{dx}{dy} = -\sin y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = -\frac{1}{\sin y}$$



$\because x = \cos y$

$\therefore \sin y = \pm\sqrt{1-\cos^2 y} = \pm\sqrt{1-x^2}$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \text{ or } \frac{1}{\sqrt{1-x^2}} \text{ (rejected)}$$

If u is a function of x, then $\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$

Given $y = \tan^{-1} x = \text{arc}\tan(x)$. Find $\frac{dy}{dx}$.

$x = \tan y$

$\frac{dx}{dy} = \sec^2 y$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

If u is a function of x, then $\frac{d}{dx}\left(\tan^{-1} u\right) = \frac{1}{1+u^2}\cdot\frac{du}{dx}$

$\frac{d(\text{arc}\sec x)}{dx} = \frac{d\left(\sec^{-1} x\right)}{dx} = \frac{1}{x\sqrt{x^2-1}}$ for $x < -1$ or $x > 1$.

$\frac{d(\text{arc}\csc x)}{dx} = \frac{d\left(\csc^{-1} x\right)}{dx} = -\frac{1}{x\sqrt{x^2-1}}$ for $x < -1$ or $x > 1$.

$\frac{d(\text{arc}\cot x)}{dx} = \frac{d\left(\cot^{-1} x\right)}{dx} = -\frac{1}{1+x^2}$

The proofs are left to you as exercises.