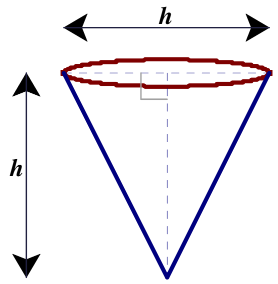
Supplementary Exercise on differentiation

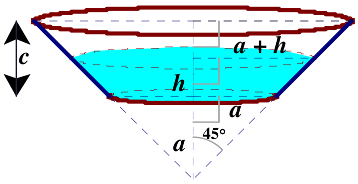
First created in 1986, retyped as MS WORD document on 20080530 by Mr. Francis Hung Last updated: 12 February 2022

1. A vessel is in the form of a hollow cone of vertical angle 60°, with vertex downwards and axis vertical. Water is poured into it at the rate of 4 cm^3/s. When the depth of water is 6 cm, at what rate is
   (a) the water rising;
   (b) the wetted surface increasing?

2. A conical funnel, whose height is equal to the diameter of its top, allows water to flow out of it through a small hole at the vertex at the rate of 0.1 cm^3/s, the axis of the funnel being vertical. At what rate is the water level descending when the depth of water in the funnel is 3 cm?



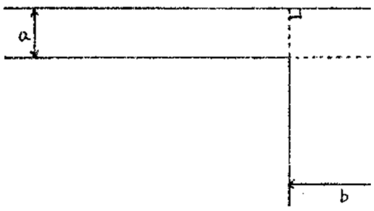
3. A vessel is in the form of a frustum of a cone of semi-vertical angle 45°. The radius of the base of the vessel is a m, the base being the smaller end. Water is poured into the vessel at the rate of b m^3/min. Find the rate at which the level of water is rising when it is c m above the base.



4. A spherical balloon is being inflated, the volume increasing at the constant rate of 15 cm^3/s. At what rate is the radius increasing when it is 10 cm long?

5. A spherical bubble is decreasing in volume at the rate of 2 cm^3/s. Find the rate at which the surface area is diminishing when the radius is 4 cm.

6. **Modified from 1985 Paper 1 Q9**
   Find the length of the longest ladder which can be carried around the corner of a corridor, whose dimensions are indicated in the figure on the right, if it is assumed that the ladder is moved parallel to the floor.

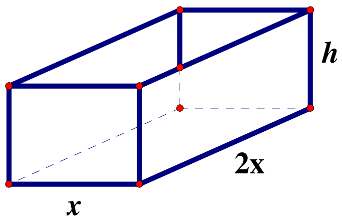


7. Find two numbers such that their sum is twelve and that the sum of the cube of one and the square of the other is a minimum. Give your answer correct to one decimal place.

8. A feeding trough is to be made from bending a long sheet of metal 80 cm wide to give a trapezoidal cross-section with sides of equal length x cm inclined at 60° to the horizontal. Find the value of x for which the cross-sectional area is the greatest.

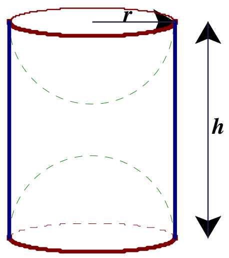
9. The distance between vehicles passing along a busy road at an average speed of v m/s is $\left(3+\frac{v}{3}+\frac{v^2}{300}\right)$ m. How many vehicles pass during an hour? What speed makes this number a maximum?

10. A metal tank is to be built in the form of a rectangular parallelepiped, open at the top and of given volume V, the sides of the base being in the ratio 2 : 1. Find its dimensions if the least area of thin sheet metal is to be used.

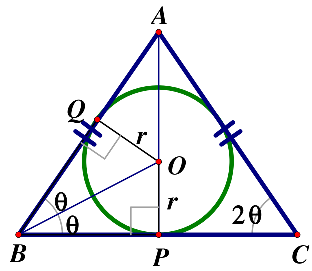


11. Prove that, as x increases from 0 to $\frac{\pi}{2}$, the function $x-\frac{3\sin x}{2+\cos x}$ continually increases.

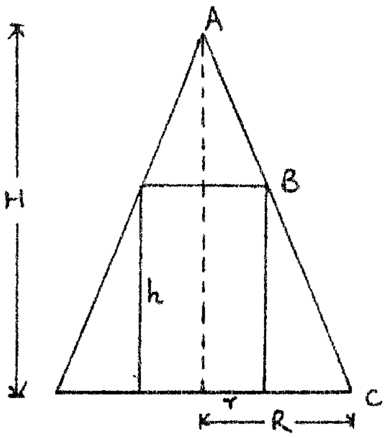
12. The plane ends of a right circular cylinder, of height h and radius r, are scooped out to form hollow hemispherical surfaces of radius r. If the volume V remaining is given, by considering $\frac{dS}{dr}$, find the value of $\frac{r}{h}$ in order that the total surface area S may be a minimum, and determine this minimum in terms of V. (Hint: First show that $S=\frac{20\pi r^2}{3}+\frac{2V}{r}$.)



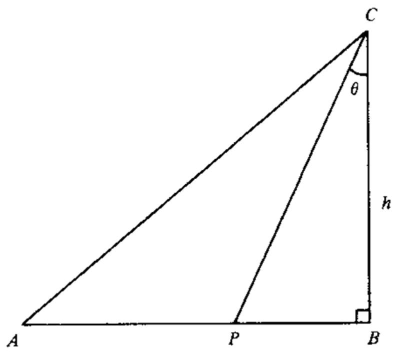
13. The figure shows a circle of centre O and radius r inscribed in a variable isosceles triangle ABC with $AB = AC$. Let $\angle ACB = 2\theta$. Prove that the area of $\Delta ABC = r^2 \cot^2 \theta \tan 2\theta$.
Hence show that the area of the triangle is a minimum (and not a maximum) when the triangle is equilateral.



14. As shown in the figure, a right circular cylinder is cut from a solid right circular cone whose axis coincides with that of the cylinder. Show that
(a) $h = H-\frac{Hr}{R}$, where H, R are the height and radius of the cone respectively, and h, r are the height and the radius of the cylinder respectively.
(b) Volume of the cylinder $V=\pi r^2\left(H-\frac{Hr}{R}\right)$.
Hence prove that the volume of the cylinder cannot exceed $\frac{4}{9}$ that of the cone.



15. **1984 Paper 1 Q11**
In the given figure, AB is a railway 50 km long. C is a factory h km from B such that $\angle ABC = 90°$. Goods are to be transported from C to A. The transportation cost per tonne of goods across the country by truck is \$2 per km, whereas by railway it is \$1 per km.
(a) Let P be a point on the railway, $\angle PCB = \theta$, and let \$$N$ be the total transportation cost for 1 tonne of goods from C to P and then to A. Find N in terms of θ and h.



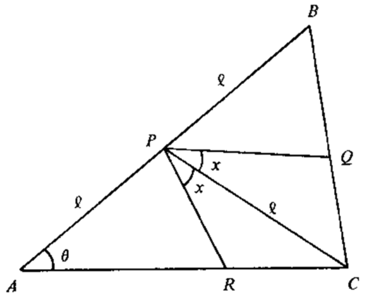
(b) If $h = 50$, show that the least transportation cost for 1 tonne of goods from C to A is \$$50(\sqrt{3}+1)$.
(c) (i) Suppose $h > 50\sqrt{3}$. Show that $\tan\theta < \frac{1}{\sqrt{3}}$, and deduce that $\frac{dN}{d\theta} < 0$ for all possible values of θ.
(ii) If $h = 200$, what route should be taken so that the transportation cost is the least?

16. ABC is a triangle in which $AB = AC$ and $\angle BAC = 2\theta$. The median $AD = h$. Find a point P on AD so that the product of the distances from P to the three sides of ΔABC is a maximum.

17. If $y = x^3 - 3x^2 + 4x$, prove that $\frac{dy}{dx}$ is positive for all real values of x.
Hence prove that y is positive for all positive real values of x.

18. **1984 Paper 2 Q11**

In the given figure, ABC is a triangle with $\angle A = \theta$. P is a point on AB such that $PA = PB = PC = \ell$. R and Q are points on AC and BC respectively, such that $\angle QPC = \angle RPC = x$.



(a) Show that $PR = \dfrac{\ell \sin\theta}{\sin(x+\theta)}$.

(b) Find $\angle PCQ$ in terms of θ and hence find PQ in terms of ℓ, x and θ.

(c) Show that the area of $\Delta PQR = \dfrac{\ell^2 \sin\theta\cos\theta\sin 2x}{2\sin(x+\theta)\cos(x-\theta)}$,

and show that it can be expressed as $\dfrac{\ell^2 \sin 2\theta}{2}\left(1-\dfrac{\sin 2\theta}{\sin 2x+\sin 2\theta}\right)$ …… (*)

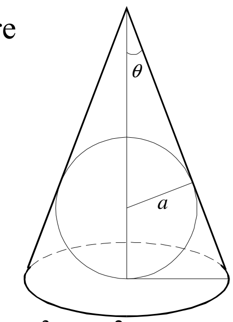
(d) (i) If $\theta = \dfrac{\pi}{8}$, find the possible range of values of x.

Hence use (*) to deduce the maximum area of ΔPQR and express it in terms of ℓ.

(ii) If $\theta = \dfrac{\pi}{12}$, what is the possible range of values of x?

Express the maximum area of ΔPQR in terms of ℓ.

19. A right circular cone of semi-vertical angle θ is circumscribed about a sphere of given radius a.



(a) Prove that the volume of the cone is $\dfrac{1}{3}\pi a^3(1+\csc\theta)^3\tan^2\theta$, and

(b) find the value of θ for which this is a minimum.

20. (a) Find the equations of the tangent and normal at (1, 1) to the curve $y = 4x^3 - 4x^2 + x$.

(b) Find the coordinates of the point in which the tangent meets the curve again.

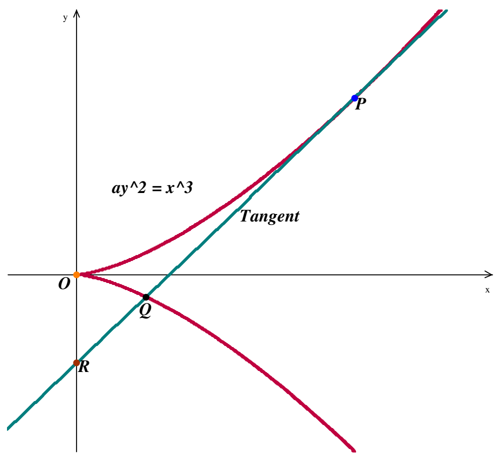
21. A curve whose equation has the form $y = x^3 + ax + b$, where a, b are constants, passes through the origin and the point (2, 6).

(a) Find the coordinates of the points where the tangent is parallel to the x-axis.

(b) Find also the equations of the tangents at the point (2, 6) and at the points where the curve meets the x-axis.

22. (a) Prove that the equation of the tangent at the point $P(4at^2, 8at^3)$ of the curve $ay^2 = x^3$ is $y = 3tx - 4at^3$.

(b) The tangent meets the curve again at Q and the y-axis at R. Show that Q is the point $(at^2, -at^3)$ and that $PQ = 3QR$.



23. A point P lies on the curve $y^2 = x^3$. The tangent at P meet the x-axis at L and the y-axis at M; the normal at P meets the x-axis at S and the y-axis at T.

(a) Find the equations of the tangent and normal at P in terms of k, where k^3 is the y-coordinate of P, and

(b) prove that $OL\cdot OS = TO\cdot OM$, where O is the origin.

24. (a) Find the equation of the tangent to the curve $3ay^2 = x^2(x + a)$ at the point $(2a, 2a)$.

(b) Find the coordinates of the point P at which this tangent meets the curve again, and

(c) prove that it is the normal to the curve at P.

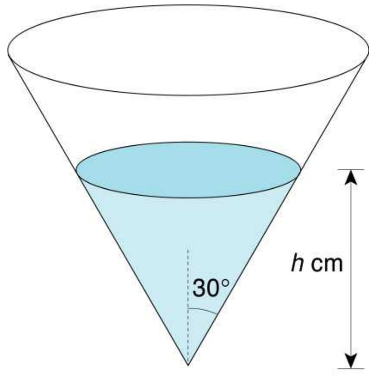


Fig. 1

1. (a) Let the depth of water be h cm.
Let the radius of surface of water be r cm.
Let the volume of water in the cone be V cm^3.

Then $r = h\tan 30° = \dfrac{h}{\sqrt{3}}$

$V = \dfrac{1}{3}\pi\left(\dfrac{h}{\sqrt{3}}\right)^2 h = \dfrac{\pi}{9}h^3$

$\dfrac{dV}{dt} = \dfrac{\pi}{3}h^2\dfrac{dh}{dt}$

When $h = 6$, $\dfrac{dV}{dt} = 4$

$4 = \dfrac{\pi}{3}(6)^2\dfrac{dh}{dt}$

$\dfrac{dh}{dt} = \dfrac{1}{3\pi}$

$\therefore$ The water is rising at $\dfrac{1}{3\pi}$ cm/s.

(b) Let the wetted surface area be S cm^2, the slant edge be L cm.

$S = \pi rL$, $L = h\sec 30° = \dfrac{2h}{\sqrt{3}}$

$S = \pi\left(\dfrac{h}{\sqrt{3}}\right)\cdot\dfrac{2h}{\sqrt{3}} = \dfrac{2\pi}{3}h^2$

$\dfrac{dS}{dt} = \dfrac{4\pi h}{3}\cdot\dfrac{dh}{dt}$

When $h = 6$, $\dfrac{dh}{dt} = \dfrac{1}{3\pi}$

$\dfrac{dS}{dt} = \dfrac{4\pi}{3}\times 6\pi\times\dfrac{1}{3\pi} = \dfrac{8}{3}$

The wetted surface is increasing at $\dfrac{8}{3}$ cm^2/s

2. Let the depth of water be h cm.
Let the volume of water in the cone be V cm^3.

$r = \dfrac{h}{2}$

$V = \dfrac{1}{3}\pi r^2 h = \dfrac{\pi}{3}\left(\dfrac{h}{2}\right)^2 h = \dfrac{\pi}{12}h^3$

$\dfrac{dV}{dt} = \dfrac{\pi}{12}\left(3h^2\right)\dfrac{dh}{dt} = \dfrac{\pi h^2}{4}\cdot\dfrac{dh}{dt}$

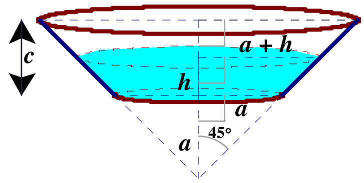
When $h = 3$, $\dfrac{dV}{dt} = -0.1$

$-0.1 = \dfrac{\pi(3)^2}{4}\cdot\dfrac{dh}{dt}$

$\dfrac{dh}{dt} = -\dfrac{4}{90\pi} = -\dfrac{2}{45\pi}$

The depth of the water level is descending at a level of $\dfrac{2}{45\pi}$ cm/s when the depth of water in the funnel is 3 cm.

3. Let the depth of water be h m at time t minutes.
The radius of surface of water is $(a + h)$ m at that time.
Let the volume of water be V m^3.



$V=\frac{\pi}{3}\left[(a+h)^3-a^3\right]=\frac{\pi}{3}\left(3a^2h+3ah^2+h^3\right)$

$\frac{dV}{dt}=\frac{\pi}{3}\left(3a^2+6ah+3h^2\right)\frac{dh}{dt}=\pi(a^2+2ah+h^2)\frac{dh}{dt}$

When $h = c$, $b=\pi(a^2+2ac+c^2)\frac{dh}{dt}$

$\frac{dh}{dt}=\frac{b}{\pi(a+c)^2}$ The rate at which the level of water is rising at $\frac{b}{\pi(a+c)^2}$ m/min.

4. Let the radius be r cm, the volume be V cm^3 at time t seconds.

$V=\frac{4\pi r^3}{3}$

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

$15=4\pi(10)^2\frac{dr}{dt}$

$\frac{dr}{dt}=\frac{3}{80\pi}$

The radius increasing at a rate of $\frac{3}{80\pi}$ cm/s.

5. Let the radius be r cm, the surface area be S cm^2, the volume be V cm^3 at time t seconds.

$V=\frac{4\pi r^3}{3}$

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

$-2=4\pi(4)^2\frac{dr}{dt}$

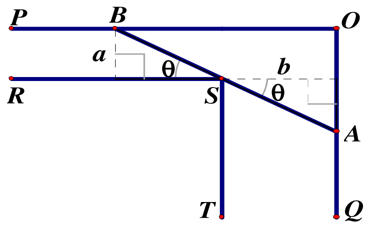
$\frac{dr}{dt}=-\frac{1}{32\pi}$

$S=4\pi r^2$

$\frac{dS}{dt}=8\pi r\frac{dr}{dt}=8\pi(4)\left(-\frac{1}{32\pi}\right)=-1$

The surface area is diminishing at the rate of 1 cm^2/s.

6. Let the ladder be AB.
Label the vertices A, B, P, O, Q, R, S, T as shown.
In order to pass through the corridor, the ladder has to be able to pass through the "narrowest" position ASB.
Suppose AB inclined at an angle θ to RS.



Let $AB = s = AS + SB$

$s = a\csc\theta + b\sec\theta$

$\frac{ds}{d\theta}=-a\csc\theta\cot\theta+b\sec\theta\tan\theta=0$

$\frac{b\sin\theta}{\cos^2\theta}=\frac{a\cos\theta}{\sin^2\theta}$

$\tan^3\theta=\frac{a}{b}$

$\tan\theta=\sqrt[3]{\frac{a}{b}}$

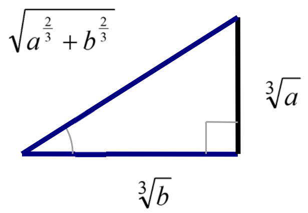
$\frac{d^2 s}{d\theta^2} = a(\csc^3\theta + \csc\theta\cot^2\theta) + b(\sec^3\theta + \sec\theta\tan^2\theta)$

$\because 0 < \theta < \frac{\pi}{2}$, $\csc\theta > 0$, $\sec\theta > 0$, $\tan\theta > 0$, $\cot\theta > 0$)

$\therefore \left.\frac{d^2 s}{d\theta^2}\right|_{\tan\theta = \sqrt[3]{\frac{a}{b}}} > 0$

$\therefore$ When $\tan\theta = \sqrt[3]{\frac{a}{b}}$, s is a minimum.

Minimum $s = a\csc\theta + b\sec\theta = \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} + \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}}$



$s = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} \sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}} = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

$\therefore$ The length of the longest ladder is $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

7. Let the two numbers be x, $12 - x$, and the sum be S.

$S = x^3 + (12 - x)^2 = x^3 + x^2 - 24x + 144$

$\frac{dS}{dx} = 3x^2 + 2x - 24 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{73}}{3}$

$\frac{d^2 S}{dx^2} = 6x + 2$

$\left.\frac{d^2 S}{dx^2}\right|_{x = \frac{-1-\sqrt{73}}{2}} = 6\times\left(\frac{-1-\sqrt{73}}{3}\right) + 2 < 0$; $\left.\frac{d^2 S}{dx^2}\right|_{x = \frac{-1+\sqrt{73}}{2}} = 6\times\left(\frac{-1+\sqrt{73}}{3}\right) + 2 > 0$

When $x = \frac{-1-\sqrt{73}}{3}$, S attains a relative maximum.

When $x = \frac{-1+\sqrt{73}}{3}$, S attains a relative minimum.

$\therefore$ The two numbers are $\frac{-1+\sqrt{73}}{3}$ and $12 - \frac{-1+\sqrt{73}}{3}$; i.e. 2.5 and 9.5 corr. to 1 d.p.

8. The width of the base of the trapezium is $(80 - 2x)$ cm.

The width of the upper base of the trapezium is

$(80 - 2x + 2x\cos 60°)$ cm $= (80 - x)$ cm

Let the area of the cross section be S cm^2.

$S = \frac{1}{2}(80 - 2x + 80 - x)\cdot x\sin 60°$

$S = \frac{\sqrt{3}}{4}(160 - 3x)x = \frac{\sqrt{3}}{4}(160x - 3x^2)$

$\frac{dS}{dx} = \frac{\sqrt{3}}{4}(160 - 6x) = 0 \Rightarrow x = \frac{80}{3}$

$\frac{d^2 S}{dx^2} = -\frac{3\sqrt{3}}{2} < 0$

$\therefore$ When $x = \frac{80}{3}$, S is a minimum

$\because$ There is only one turning point

$\therefore S$ attains the absolute maximum.

9. Suppose Car A passes a certain point O at time $t = 0$ sec.

After t sec, another Car B passes the point O. Car A has moved a distance $\left(3+\frac{v}{3}+\frac{v^2}{300}\right)$ m.

$$t=\frac{\text{distance}}{\text{speed}}=\frac{\text{separation between A and B}}{\text{average speed}}=\frac{1}{v}\left(3+\frac{v}{3}+\frac{v^2}{300}\right)$$

In one hour, suppose there are y cars passing O.

$$y=\frac{60\times 60}{\frac{3}{v}+\frac{1}{3}+\frac{v}{300}}=\frac{1080000v}{900+100v+v^2}$$

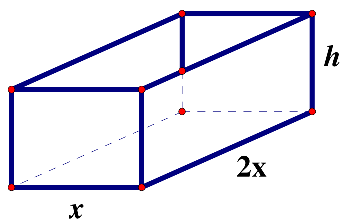
$$\frac{\mathrm{d}y}{\mathrm{d}v}=\frac{1080000\left[v^2+100v+900-v(100+2v)\right]}{\left(v^2+100v+900\right)^2}=\frac{1080000\left(900-v^2\right)}{\left(v^2+100v+900\right)^2}=0\Rightarrow v=30$$

v	30^-	30	30^+
$\frac{\mathrm{d}y}{\mathrm{d}v}$	+	0	–

$\therefore$ When $v = 30$, y is a relative maximum

The speed is 30 m/s.

10. Let the width of the box be x, length be $2x$ and height be h.



$$V=x(2x)h\Rightarrow h=\frac{V}{2x^2}$$

Let the total area be S.

$S = 2xh + 2(2xh) + 2x^2 = 6xh + 2x^2$

$$S=6x\cdot\frac{V}{2x^2}+2x^2=\frac{3V}{x}+2x^2$$

$$\frac{\mathrm{d}S}{\mathrm{d}x}=-\frac{3V}{x^2}+4x=0\Rightarrow 4x^3=3V\Rightarrow x=\left(\frac{3V}{4}\right)^{1/3}$$

$$\frac{\mathrm{d}^2S}{\mathrm{d}x^2}=\frac{6V}{x^3}+4>0 \text{ for all } x>0$$

$\therefore$ When $x=\left(\frac{3V}{4}\right)^{1/3}$, the area S is a relative minimum.

$\because$ There is only one turning point, it is an absolute minimum.

The other two sides are $2x$, h.

$$2x=2\left(\frac{3V}{4}\right)^{1/3}=2^{1/3}(3V)^{1/3}=(6V)^{1/3},$$

$$h=\frac{V}{2x^2}=\frac{V}{2\left(\frac{3V}{4}\right)^{2/3}}=\left(\frac{2V}{9}\right)^{1/3}$$

11. $y=x-\frac{3\sin x}{2+\cos x}$

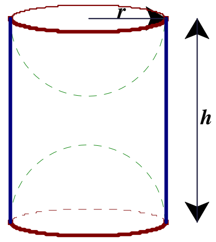
$$\frac{\mathrm{d}y}{\mathrm{d}x}=1-3\cdot\frac{(2+\cos x)\cos x+\sin^2 x}{(2+\cos x)^2}=1-3\cdot\frac{1+2\cos x}{(2+\cos x)^2}=1-3\cdot\frac{4+2\cos x-3}{(2+\cos x)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{6}{2+\cos x}+\frac{9}{(2+\cos x)^2}=\left(1-\frac{3}{2+\cos x}\right)^2\geq 0 \text{ for all } x$$

$\therefore$ y is an increasing function.

$\therefore$ As x increases from 0 to $\frac{\pi}{2}$, the function $x-\frac{3\sin x}{2+\cos x}$ continually increases.

12. $V = \pi r^2 h - \dfrac{4\pi r^3}{3}$



$h = \dfrac{V + \dfrac{4\pi r^3}{3}}{\pi r^2}$

$S = 2\pi rh + 4\pi r^2$

$S = 2\pi r\left(\dfrac{V + \dfrac{4}{3}\pi r^3}{\pi r^2}\right) + 4\pi r^2$

$S = \dfrac{2V}{r} + \dfrac{8\pi r^2}{3} + 4\pi r^2 = \dfrac{2V}{r} + \dfrac{20\pi r^2}{3}$

$\dfrac{\mathrm{d}S}{\mathrm{d}r} = \dfrac{40\pi r}{3} - \dfrac{2V}{r^2} = 0 \Rightarrow r^3 = \dfrac{3V}{20\pi}$

$\dfrac{\mathrm{d}^2S}{\mathrm{d}r^2} = \dfrac{40\pi}{3} + \dfrac{4V}{r^3} > 0$ for all $r > 0$

$\therefore S$ is a relative minimum when $r^3 = \dfrac{3V}{20\pi}$

$h = \dfrac{V + \dfrac{4\pi}{3}\left(\dfrac{3V}{20\pi}\right)}{\pi r^2}$

$\dfrac{h}{r} = \dfrac{V\left(1 + \dfrac{1}{5}\right)}{\pi r^3} = \dfrac{6V}{5} \div \dfrac{3V}{20} = 8$

$\therefore$The ratio $\dfrac{r}{h} = \dfrac{1}{8}$

$S = \dfrac{2V}{r} + \dfrac{20\pi r^2}{3} = \dfrac{2V}{\left(\dfrac{3V}{20\pi}\right)^{1/3}} + \dfrac{20\pi\left(\dfrac{3V}{20\pi}\right)^{2/3}}{3} = \dfrac{(20\pi)^{1/3} V^{2/3}(1+2)}{3^{1/3}} = (20\pi \times 3^2 V^2)^{1/3} = (180\pi V^2)^{1/3}$

13. Suppose the circle touches BC at P and AB at Q.

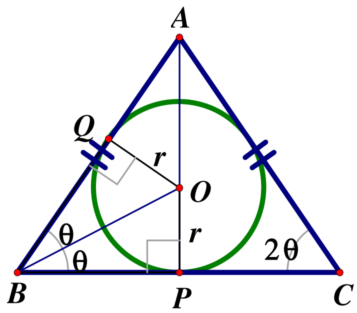
Join BO, then $\angle OBP = \angle OBQ = \theta$ (tangent from ext. pt.)

$\angle BPO = 90° = \angle BQO$ (tangent $\perp$ radius)

$BP = r \cot\theta = PC \Rightarrow BC = 2r\cot\theta$

In ΔABP, $AB = BP \sec 2\theta = r\cot\theta \sec 2\theta$

Let the area of the triangle ABC be S.



$S = \dfrac{1}{2} BC \cdot AB \sin 2\theta$

$S = \dfrac{1}{2} 2r\cot\theta \cdot (r\cot\theta \sec 2\theta)\sin 2\theta$

$S = r^2 \cot^2\theta \tan 2\theta$

$\dfrac{\mathrm{d}S}{\mathrm{d}\theta} = 2r^2 \cot\theta(-\csc^2\theta \tan 2\theta + \cot\theta \sec^2 2\theta)$

Let $\dfrac{\mathrm{d}S}{\mathrm{d}\theta} = 0 \Rightarrow \cot\theta = 0$ or $\tan 2\theta \csc^2\theta = \cot\theta \sec^2 2\theta$

$\theta = \dfrac{\pi}{2}$ or $\dfrac{\sin 2\theta}{\cos 2\theta} \cdot \dfrac{1}{\sin^2\theta} = \dfrac{\cos\theta}{\sin\theta} \cdot \dfrac{1}{\cos^2 2\theta}$

$\theta = \dfrac{\pi}{2}$ or $\sin 2\theta \cos 2\theta = \sin\theta\cos\theta$

$\theta = \dfrac{\pi}{2}$ or $2\sin 2\theta \cos 2\theta = 2\sin\theta\cos\theta$

$\theta = \frac{\pi}{2}$ or $\sin 4\theta = \sin 2\theta$

$\theta = \frac{\pi}{2}$, $4\theta = 2\theta$ or $4\theta = \pi - 2\theta$

$\theta = \frac{\pi}{2}$ (rejected), $\theta = 0$ (rejected) or $\theta = \frac{\pi}{6}$

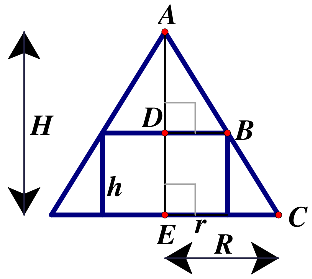
$\frac{d^2S}{d\theta^2} = 2r^2(\csc^4\theta \tan 2\theta - 4\csc^2\theta \cot\theta \sec^2 2\theta + 2\csc^2\theta \cot^2\theta \tan 2\theta + 4\cot^2\theta \sec^2 2\theta \tan 2\theta)$

$\left.\frac{d^2S}{d\theta^2}\right|_{\theta=\frac{\pi}{6}} = 2r^2\left[2^4\sqrt{3} - 4\cdot 2^2\sqrt{3}\cdot 2^2 + 2\cdot 2^2\left(\sqrt{3}\right)^2\cdot\sqrt{3} + 4\cdot\left(\sqrt{3}\right)^2\cdot 2^2\cdot\sqrt{3}\right] = 48\sqrt{3}r^2 > 0$

$\therefore$ When $\theta = \frac{\pi}{6}$, S attains the minimum.

$2\theta = \frac{\pi}{3}$, $\angle B = \angle C = \frac{\pi}{3}$ $\therefore \Delta ABC$ is equilateral.

14. Mark the points D and E as shown.



$\angle ADB = \angle AEC = 90°$; $DB = r$, $AD = H - h$, $DE = h$

By similar triangles: $\frac{H}{R} = \frac{H-h}{r}$

$\frac{rH}{R} = H - h$

$h = H - \frac{rH}{R}$

Volume of the cylinder $V = \pi r^2 h$

$V = \pi r^2\left(H - \frac{Hr}{R}\right)$, H and R are constants

$V = \pi H\left(r^2 - \frac{r^3}{R}\right)$

$\frac{dV}{dr} = \pi H\left(2r - \frac{3r^2}{R}\right) = 0 \Rightarrow r = 0$ (rejected) or $2R = 3r$; i.e. $r = \frac{2R}{3}$

$\frac{d^2V}{dr^2} = \pi H\left(2 - \frac{6r}{R}\right)$

$\left.\frac{d^2V}{dr^2}\right|_{r=\frac{2R}{3}} = -2\pi H < 0$

$\therefore$When $r = \frac{2R}{3}$, the volume of the cylinder V is a maximum.

The maximum volume $V = \pi H\left[\left(\frac{2R}{3}\right)^2 - \frac{1}{R}\left(\frac{2R}{3}\right)^3\right] = \frac{4\pi HR^2}{27}$

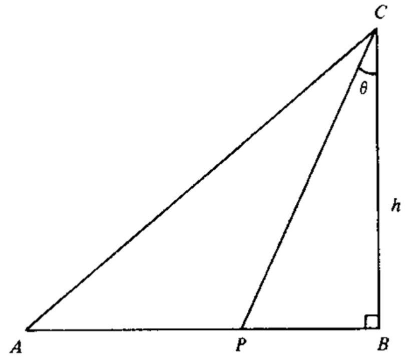
Max. $V = \frac{4}{9}\left(\frac{\pi HR^2}{3}\right) = \frac{4}{9} \times$ volume of the cone.

$\therefore$ The volume of the cylinder cannot exceed $\frac{4}{9}$ that of the cone.

15. (a) $N = 2CP + AP$

$= 2h\sec\theta + 50 - PB$

$= 2h\sec\theta + 50 - h\tan\theta$



(b) $N = 50 + h(2\sec\theta - \tan\theta)$

$\frac{dN}{d\theta} = h(2\sec\theta\tan\theta - \sec^2\theta) = 0$

$\sec\theta = 0$ (rejected) or $2\tan\theta = \sec\theta$

$2\sin\theta = 1$

$\theta = \frac{\pi}{6}$

$\frac{d^2N}{d\theta^2} = h(2\sec\theta\tan^2\theta + 2\sec^3\theta - 2\sec^2\theta\tan\theta)$

$\left.\frac{d^2N}{d\theta^2}\right|_{\frac{\pi}{6}} = h\left(2\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{3} + 2\cdot\frac{8}{3\sqrt{3}} - 2\cdot\frac{4}{3}\cdot\frac{1}{\sqrt{3}}\right) = \frac{4h}{\sqrt{3}} > 0$

$\therefore$ When $\theta = \frac{\pi}{6}$, N attains the minimum.

When $h = 50$, $\theta = \frac{\pi}{6}$, $N = 50\left(1 + 2\times\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) = 50\left(1+\sqrt{3}\right)$

$\therefore$ The least transportation cost for 1 tonne of goods from C to A is $\$50\left(1+\sqrt{3}\right)$.

(c) (i) When $h > 50\sqrt{3}$, $\frac{50}{h} < \frac{1}{\sqrt{3}}$

$\theta < \angle ACB$

$\tan\theta < \tan\angle ACB = \frac{AB}{BC}$

$\tan\theta < \frac{50}{h} < \frac{1}{\sqrt{3}}$

$\tan\theta < \frac{1}{\sqrt{3}}$

$0 \le \theta < \frac{\pi}{6}$

$\frac{dN}{d\theta} = h(2\sec\theta\tan\theta - \sec^2\theta) = h\sec^2\theta\,(2\sin\theta - 1)$

Clearly $\sec^2\theta > 0$, $0 \le \sin\theta < \sin\frac{\pi}{6} = \frac{1}{2}$

$\therefore 2\sin\theta < 1 \Rightarrow 2\sin\theta - 1 < 0$

$\therefore \frac{dN}{d\theta} < 0$ for all possible values of θ.

(ii) $h = 200$, $\tan\angle ACB = \frac{50}{200} = \frac{1}{4}$

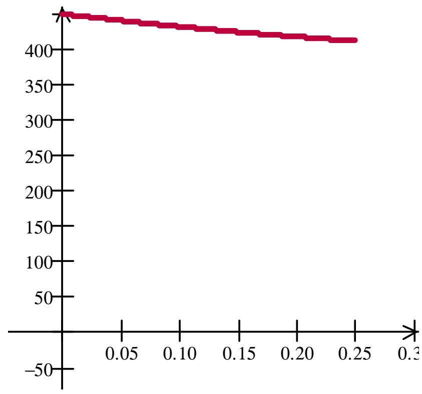
$\Rightarrow 0 \le \theta \le \tan^{-1}\left(\frac{1}{4}\right)$

From $(c)(i)$, $\frac{dN}{d\theta} < 0$

N is decreasing.

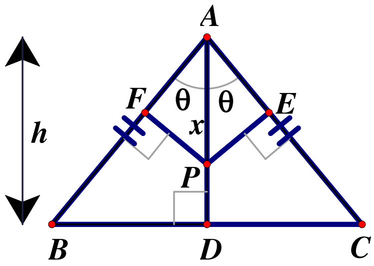
From the graph, when $\theta = \tan^{-1}\left(\frac{1}{4}\right)$

N attains the absolute minimum.



The route should be taken as CA (directly) so that the transportation cost is the least.

16. $\because AD$ = median



$\therefore BD = CD$ and $AD \perp BC$

Let E and F be the feet of perpendiculars from P onto AC and AB respectively, let $AP = x$, $PD = h - x$, $\angle BAD = \angle CAD = \theta$

By symmetry, $PE = PF = x \sin\theta$

Let y = product of distances = $PD \cdot PE \cdot PF$

$y = (x\sin\theta)^2 (h - x)$

$y = \sin^2\theta(hx^2 - x^3)$; h, θ are constants, x is a variable

$\frac{dy}{dx} = \sin^2\theta(2hx - 3x^2) = 0 \Rightarrow x = 0$ (rejected) or $x = \frac{2h}{3}$

$\frac{d^2y}{dx^2} = 2\sin^2\theta(h - 3x)$; $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{2h}{3}} < 0$

$\therefore$ When $x = \frac{2h}{3}$ $\frac{2h}{3}$, y is a relative maximum.

So P divides AD in the ratio 2 : 1, in this case P is the centriod.

17. $y = x^3 - 3x^2 + 4x$

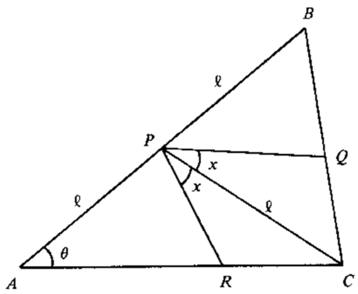
$\frac{dy}{dx} = 3x^2 - 6x + 4 = 3(x - 1)^2 + 1 > 0$ for all x.

$\therefore y$ is strictly increasing for all x.

When $x > 0$, $x^3 - 3x^2 + 4x > 0^3 - 3\times0^2 + 4\times0 = 0$

$\therefore$ y is positive for all positive real values of x.

18. (a) $PA = PC \Rightarrow \angle PCA = \theta$



$\therefore \angle PRA = x + \theta$ (ext. $\angle$ of Δ)

In ΔPRA, $\frac{PR}{\sin\theta} = \frac{\ell}{\sin(x+\theta)}$

$\therefore PR = \frac{\ell\sin\theta}{\sin(x+\theta)}$

(b) $PC = PB \Rightarrow \angle PCQ = \angle PBQ$ $(= \phi)$

$2(\theta + \phi) = \pi \Rightarrow \phi = \frac{\pi}{2} - \theta$ ($\angle$ sum of ΔABC)

$\therefore \angle PCQ = \phi = \frac{\pi}{2} - \theta$

$\therefore \angle PQB = x + \phi$ (ext. $\angle$ of Δ)

In ΔPQB, $\frac{PQ}{\sin\phi} = \frac{\ell}{\sin(x+\phi)}$

$$\therefore PQ = \frac{\ell\sin\phi}{\sin(x+\phi)} = \frac{\ell\sin\left(\frac{\pi}{2}-\theta\right)}{\sin\left(x+\frac{\pi}{2}-\theta\right)}$$

$$PQ = \frac{\ell\cos\theta}{\cos(\theta - x)} = \frac{\ell\cos\theta}{\cos(x-\theta)}$$

(c) Area of $\Delta PQR = \frac{1}{2} PQ \cdot PR \sin 2x$

$$= \frac{\ell^2\sin\theta\cos\theta\sin 2x}{2\sin(x+\theta)\cos(x-\theta)} = \frac{\ell^2}{2}\cdot\frac{\sin 2\theta\sin 2x}{\sin 2x + \sin 2\theta}$$

$$= \frac{\ell^2\sin 2\theta}{2}\cdot\left(\frac{\sin 2x + \sin 2\theta - \sin 2\theta}{\sin 2x + \sin 2\theta}\right)$$

$$= \frac{\ell^2\sin 2\theta}{2}\cdot\left(1 - \frac{\sin 2\theta}{\sin 2x + \sin 2\theta}\right) \quad \cdots\cdots (*)$$

(d) (i) Let $\theta = \frac{\pi}{8}$

$\phi = \frac{\pi}{2} - \theta = \frac{3\pi}{8}$

$0 < x \le \pi - 2\theta$ and $0 < x \le \pi - 2\phi$

$0 < x \le \frac{\pi}{4}$

$0 < \sin 2x \le 1$

The maximum area of ΔPQR is

$\frac{\ell^2 \sin 2\theta}{2} \cdot \left(1 - \frac{\sin 2\theta}{1 + \sin 2\theta}\right) = \frac{\ell^2}{2(1+\sqrt{2})} = \frac{\ell^2(\sqrt{2}-1)}{2} = 0.207\ell^2$

(ii) If $\theta = \frac{\pi}{12}$, then $\phi = \frac{5\pi}{12}$ and $0 < x \le \frac{\pi}{6}$

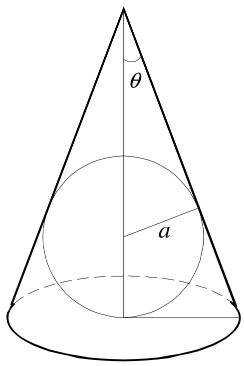
The maximum area of ΔPQR

$= \frac{\ell^2 \sin\frac{\pi}{6}}{2} \cdot \left(1 - \frac{\sin\frac{\pi}{6}}{\sin\frac{\pi}{3} + \sin\frac{\pi}{6}}\right) = \frac{\ell^2}{4}\left(1 - \frac{1}{\sqrt{3}+1}\right) = \frac{\ell^2\sqrt{3}}{4(\sqrt{3}+1)} = \frac{\ell^2\sqrt{3}(\sqrt{3}-1)}{8} = 0.158\ell^2$

19. (a) Height of the cone = $h = a + a \csc\theta$

Base radius of the cone = $h \tan\theta$

$= (a + a\csc\theta)\tan\theta$



$\therefore V = \frac{1}{3}\pi\,[(a + a\csc\theta)\tan\theta]^2\,(a + a\csc\theta)$

$= \frac{1}{3}\pi a^3 (1+\csc\theta)^3 \tan^2\theta$

(b) $\frac{dV}{d\theta} = \frac{\pi a^3}{3}\left[3(1+\csc\theta)^2(-\csc\theta\cot\theta)\tan^2\theta + (1+\csc\theta)^3(2\tan\theta\sec^2\theta)\right]$

$= \frac{\pi a^3}{3}(1+\csc\theta)^2\tan\theta\,[-3\csc\theta + 2(1+\csc\theta)\sec^2\theta]$

$= \frac{\pi a^3}{3}(1+\csc\theta)^2 \cdot \frac{\sin\theta}{\cos\theta}\left[\frac{-3}{\sin\theta} + 2\left(1+\frac{1}{\sin\theta}\right)\cdot\frac{1}{\cos^2\theta}\right]$

$= \frac{\pi a^3}{3}(1+\csc\theta)^2\left(\frac{-3\cos^2\theta + 2\sin\theta + 2}{\cos^3\theta}\right)$

$= \frac{\pi a^3}{3}(1+\csc\theta)^2\left(\frac{3\sin^2\theta + 2\sin\theta - 1}{\cos^3\theta}\right)$

$= \frac{\pi a^3}{3}(1+\csc\theta)^2 \cdot \frac{(3\sin\theta - 1)(\sin\theta + 1)}{\cos^3\theta}$

When $0 < \theta < \sin^{-1}\left(\frac{1}{3}\right)$, $\frac{dV}{d\theta} < 0$; when $\sin^{-1}\left(\frac{1}{3}\right) < \theta < \frac{\pi}{2}$, $\frac{dV}{d\theta} > 0$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, $\frac{dV}{d\theta} = 0$

$\therefore$ V attains the minimum when $\theta = \sin^{-1}\left(\frac{1}{3}\right) = 0.3398$.

20. (a) $y = 4x^3 - 4x^2 + x$

When $x = 1$, $y = 4 - 4 + 1 = 1$; $\therefore$ The point (1, 1) lies on the curve.

$\frac{dy}{dx} = 12x^2 - 8x + 1$

$\left.\frac{dy}{dx}\right|_{x=1} = 12 - 8 + 1 = 5$

Equation of tangent is: $y - 1 = 5(x - 1) \Rightarrow 5x - y - 4 = 0$

Equation of normal: $y - 1 = -\frac{1}{5}(x - 1) \Rightarrow x + 5y - 6 = 0$

(b) $\begin{cases} y = 4x^3 - 4x^2 + x \\ y = 5x - 4 \end{cases}$

$4x^3 - 4x^2 + x = 5x - 4$

$4x^3 - 4x^2 - 4x + 4 = 0$

$x^2(x - 1) - (x - 1) = 0$

$(x^2 - 1)(x - 1) = 0 \Rightarrow (x - 1)^2 (x + 1) = 0$

$x = 1$ (rejected) or $x = -1$; $y = 5(-1) - 4 = -9$

$\therefore$ The coordinates of the point in which the tangent meets the curve again is (–1, –9).

21. (a) The curve $y = x^3 + ax + b$ passes through (0, 0) and (2, 6).

$\therefore 0 = b$ and $6 = 8 + 2a + b$

$\therefore a = -1$ and $b = 0$

$y = x^3 - x$

$\frac{dy}{dx} = 3x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

When $x = \frac{1}{\sqrt{3}}$, $y = \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}} = -\frac{2}{3\sqrt{3}}$;

When $x = -\frac{1}{\sqrt{3}}$, $y = \left(-\frac{1}{\sqrt{3}}\right)^3 + \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}$.

Two points $= \left(\frac{1}{\sqrt{3}}, -\frac{2}{3\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$ whose tangents are parallel to x-axis

(b) $\left.\frac{dy}{dx}\right|_{x=2} = 3\times2^2 - 1 = 11$

Equation of tangent at (2, 6) is $y - 6 = 11(x - 2)$

$11x - y - 16 = 0$

Let $y = x^3 - x = 0$

$x = 0$ or ±1

$\left.\frac{dy}{dx}\right|_{x=0} = -1$; $\left.\frac{dy}{dx}\right|_{x=1} = 2$; $\left.\frac{dy}{dx}\right|_{x=-1} = 2$

Equation of tangent at $x = 0$ is $y = -x \Rightarrow x + y = 0$

Equation of tangent at $x = 1$ is $y = 2(x - 1) \Rightarrow 2x - y - 2 = 0$

Equation of tangent at $x = -1$ is $y = 2(x + 1) \Rightarrow 2x - y + 2 = 0$

22. (a) Differentiate $ay^2 = x^3$ with respect to x.

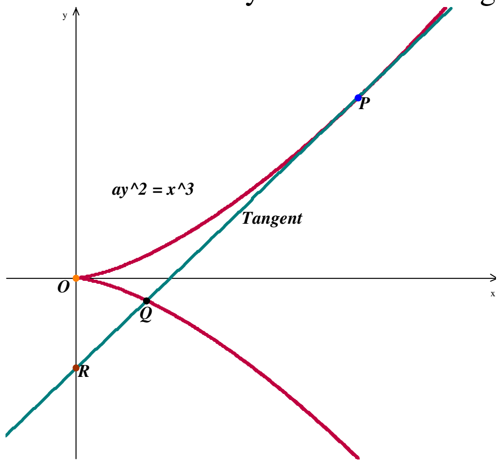
$2ay\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{2ay}$

$\left.\frac{dy}{dx}\right|_{(4at^2, 8at^3)} = \frac{3(4at^2)^2}{2a(8at^3)} = 3t$

Equation of tangent at this point is: $y - 8at^3 = 3t(x - 4at^2) \Rightarrow y = 3tx - 4at^3$

(b) $\begin{cases} ay^2 = x^3 \\ y = 3tx - 4at^3 \end{cases}$



$a(3tx - 4at^3)^2 = x^3$

$9at^2x^2 - 24a^2t^4x + 16a^3t^6 = x^3$

$x^3 - 9at^2x^2 + 24a^2t^4x - 16a^3t^6 = 0$

By division, $(x - 4at^2)(x^2 - 5at^2x + 4a^2t^4) = 0$

$(x - 4at^2)^2(x - at^2) = 0$

$x = 4at^2$ (rejected) or $x = at^2$

When $x = at^2$, $y = 3t(at^2) - 4at^3 = -at^3$

The tangent meets the curve again at Q $(at^2, -at^3)$

Let $x = 0$ in the tangent $y = 3tx - 4at^3$

$y = -4at^3$

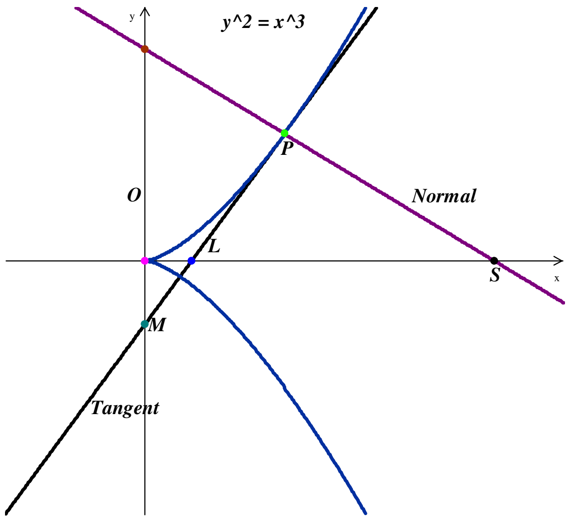
$\Rightarrow R(0, -4at^3)$

Let $PQ : QR = r : 1$

The x-coordinate of Q: $at^2 = \dfrac{4at^2}{r+1} \Rightarrow r = 3$

$\therefore PQ = 3QR$

23. (a) The y-coordinate of P be k^3.



Sub. into $y^2 = x^3 \Rightarrow (k^3)^2 = x^3$

$x = k^2 \Rightarrow P(k^2, k^3)$

Differentiate $y^2 = x^3$ with respect to x.

$2y\dfrac{dy}{dx} = 3x^2$

$\dfrac{dy}{dx} = \dfrac{3x^2}{2y} = \dfrac{3(k^2)^2}{2(k^3)} = \dfrac{3k}{2}$

Equation of tangent: $y - k^3 = \dfrac{3k}{2}(x - k^2)$

Equation of normal: $y - k^3 = -\dfrac{2}{3k}(x - k^2)$

(b) To find L: $0 - k^3 = \dfrac{3k}{2}(x - k^2) \Rightarrow x = \dfrac{k^2}{3}$ $\therefore L = \left(\dfrac{k^2}{3}, 0\right)$

To find M: $y - k^3 = \dfrac{3k}{2}(0 - k^2)$

$y = -\dfrac{k^3}{2}$ $\therefore M = \left(0, -\dfrac{k^3}{2}\right)$

To find S: $0 - k^3 = -\dfrac{2}{3k}(x - k^2)$

$x = k^2 + \dfrac{3k^4}{2}$ $\therefore S = \left(k^2 + \dfrac{3k^4}{2}, 0\right)$

To find T: $y - k^3 = -\dfrac{2}{3k}(0 - k^2)$

$y = k^3 + \dfrac{2k}{3}$ $\therefore T = \left(0, k^3 + \dfrac{2k}{3}\right)$ $\left(0, k^3 + \dfrac{2k}{3}\right)$

$OL \cdot OS = \dfrac{k^2}{3} \cdot \left(k^2 + \dfrac{3k^4}{2}\right) = \dfrac{k^4}{6} \cdot (2 + 3k^2)$

$TO \cdot OM = \left(k^3 + \dfrac{2k}{3}\right) \cdot \dfrac{k^3}{2} = \dfrac{k^4}{6} \cdot (2 + 3k^2)$

$\therefore OL \cdot OS = TO \cdot OM$

24. (a) Put $y = 2a$ into LHS = $3a(2a)^2 = 12a^3$

Put $x = 2a$ into RHS = $(2a)^2(2a + a) = 12a^3$

$\therefore$ $(2a, 2a)$ lies on the curve $3ay^2 = x^2(x + a)$

$3ay^2 = x^3 + ax^2$ ⋯⋯ (1)

Differentiate with respect to x

$6ay\frac{dy}{dx} = 3x^2 + 2ax$

$\frac{dy}{dx} = \frac{3x^2 + 2ax}{6ay}$

$\left.\frac{dy}{dx}\right|_{(2a,2a)} = \frac{3(2a)^2 + 2a(2a)}{6a(2a)} = \frac{4}{3}$

Equation of tangent: $y - 2a = \frac{4}{3}(x - 2a)$

$3y - 6a = 4x - 8a$

$4x - 3y = 2a$

(b) $y = \frac{1}{3}(4x - 2a)$ ⋯⋯ (2)

Sub. (2) into (1): $3a\left[\frac{1}{3}(4x - 2a)\right]^2 = x^3 + ax^2$

$16ax^2 - 16a^2x + 4a^3 = 3x^3 + 3ax^2$

$3x^3 - 13ax^2 + 16a^2x - 4a^3 = 0$

By division, $(x - 2a)(3x^2 - 7ax + 2a^2) = 0$

$(x - 2a)^2(3x - a) = 0$

$x = 2a$ (rejected) or $x = \frac{a}{3}$, $y = -\frac{2a}{9}$

The coordinates of P at which this tangent meets the curve again is $\left(\frac{a}{3}, -\frac{2a}{9}\right)$.

(c) $\left.\frac{dy}{dx}\right|_{\left(\frac{a}{3}, -\frac{2a}{9}\right)} = \frac{3\left(\frac{a}{3}\right)^2 + 2a\left(\frac{a}{3}\right)}{6a\left(-\frac{2a}{9}\right)} = -\frac{3}{4}$

$\therefore$ Slope of normal at $\left(\frac{a}{3}, -\frac{2a}{9}\right)$ is $\frac{4}{3}$

Equation of normal at $\left(\frac{a}{3}, -\frac{2a}{9}\right)$ is: $y + \frac{2a}{9} = \frac{4}{3}\left(x - \frac{a}{3}\right)$

$9y + 2a = 12x - 4a$

$4x - 3y = 2a$

The equation of tangent at $(2a, 2a)$ is the same as the normal to the curve at $\left(\frac{a}{3}, -\frac{2a}{9}\right)$.