Tangent Inflexion

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Breakthrough Calculus and Coordinate Geometry by Y.L. Ng & K.M. Pang p.211 Q8

Given a curve $y = e^{-\frac{(x-a)^2}{2}}$, ($a > 0$).

If two distinct tangents can be drawn from the origin to the curve,

(i) show that $a > 2$.

(ii) show that, between the two points of contact P and Q, there is only one point of inflexion.

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = -(x-a)e^{-\frac{(x-a)^2}{2}} = (a-x)e^{-\frac{(x-a)^2}{2}}$

Suppose the point of contact is (x_0, y_0),

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_0, y_0)} = (a-x_0)e^{-\frac{(x_0-a)^2}{2}}$$

Equation of tangent: $\frac{y-y_0}{x-x_0} = (a-x_0)e^{-\frac{(x_0-a)^2}{2}}$

$\because$ It passes through the origin,

$y_0 = x_0(a-x_0)e^{-\frac{(x_0-a)^2}{2}}$

$e^{-\frac{(x_0-a)^2}{2}} = x_0(a-x_0)e^{-\frac{(x_0-a)^2}{2}}$

$1 = x_0(a - x_0)$

$x_0^2 - ax_0 + 1 = 0$ (*)

For distinct real roots, $\Delta > 0$.

$a^2 - 4 > 0$

$(a + 2)(a - 2) > 0$

$a > 2$ ($\because$ given that $a > 0$)

(ii) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -e^{-\frac{(x-a)^2}{2}} - (a-x)(x-a)e^{-\frac{(x-a)^2}{2}}$

$= \left[(x-a)^2 - 1\right]e^{-\frac{(x-a)^2}{2}}$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (x-a+1)(x-a-1)e^{-\frac{(x-a)^2}{2}}$ (**)

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \Rightarrow x = a - 1$ or $a + 1$

Solve (*): $x = \frac{a+\sqrt{a^2-4}}{2}$ or $x = \frac{a-\sqrt{a^2-4}}{2}$

$$a+1-\frac{a+\sqrt{a^2-4}}{2}=\frac{a+2-\sqrt{a^2-4}}{2}>0$$

$$a+1>\frac{a+\sqrt{a^2-4}}{2}$$

$$a-1-\frac{a-\sqrt{a^2-4}}{2}=\frac{a-2+\sqrt{a^2-4}}{2}>0$$

$$a-1>\frac{a-\sqrt{a^2-4}}{2}$$

$$\frac{a+\sqrt{a^2-4}}{2}-(a-1)=\frac{\sqrt{a^2-4}-a+2}{2}$$

$$=\frac{\sqrt{a^2-4}^2-(a-2)^2}{2\left(\sqrt{a^2-4}+a-2\right)}$$

$$=\frac{4a-8}{2\left(\sqrt{a^2-4}+a-2\right)}>0$$

$$\therefore\ \frac{a-\sqrt{a^2-4}}{2}<a-1<\frac{a+\sqrt{a^2-4}}{2}<a+1$$

$\therefore$ There is at most one point ($x = a - 1$) between the two points of contact P ($x = \frac{a-\sqrt{a^2-4}}{2}$) and Q ($x = \frac{a+\sqrt{a^2-4}}{2}$) at which $\frac{d^2y}{dx^2}=0$

Consider (**), when $x < a - 1$, $\frac{d^2y}{dx^2}<0$; when $x > a - 1$, $\frac{d^2y}{dx^2}>0$.

f(x) changes from concave to convex , f($a - 1$) is a point of inflexion.