Integration formulae

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Integration is the inverse process of differentiation.

If $\frac{\mathrm{d}F(x)}{\mathrm{d}x} = \mathrm{f}(x)$, $F(x)$ is the primitive function, f(x) is the derivative of $F(x)$, then

$\int \mathrm{f}(x)\mathrm{d}x = F(x)$, f($x$) is the integrand, $F(x)$ is the primitive function, $\int$ is the integral sign.

Read $\int \mathrm{f}(x)\mathrm{d}x = F(x)$ as 'Integrate f(x) dx is equal to $F(x)$.'

Example If $\frac{\mathrm{d}(x^2+3x)}{\mathrm{d}x} = 2x+3$, then $\int (2x+3)\mathrm{d}x = x^2+3x$

However, $\frac{\mathrm{d}(x^2+3x+\pi)}{\mathrm{d}x} = 2x+3$, and $\frac{\mathrm{d}\left(x^2+3x-\frac{1}{2}\right)}{\mathrm{d}x} = 2x+3$

Therefore, $\int (2x+3)\mathrm{d}x = x^2+3x+\pi$ and $\int (2x+3)\mathrm{d}x = x^2+3x+\frac{1}{2}$

In general, we add a constant C after the primitive function. e.g. $\int (2x+3)\mathrm{d}x = x^2+3x+C$

Law of indefinite integrals

(A) $\int \mathrm{d}x = x + C$

(B) If k is a constant, $\int k\mathrm{f}(x)\mathrm{d}x = k\int \mathrm{f}(x)\mathrm{d}x$. e.g. $\int 7(2x+3)\mathrm{d}x = 7(x^2+3x)+C$

(C) If f(x) and $g(x)$ are integrable function, then $\int [\mathrm{f}(x)+\mathrm{g}(x)]\mathrm{d}x = \int \mathrm{f}(x)\mathrm{d}x + \int \mathrm{g}(x)\mathrm{d}x$.

(D) If n is real number $\neq -1$, then $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$. e.g. $\int x^{3.5}\mathrm{d}x = \frac{2x^{4.5}}{9} + C$

(E) $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$

(F) $\int \mathrm{e}^x \mathrm{d}x = e^x + C$ and $\int \mathrm{e}^{ax}\mathrm{d}x = \frac{1}{a}e^{ax} + C$ for any non-zero constant a.

(G) If $a > 0$ and $a \neq 0$ and $a \neq 1$, then $\int a^x \mathrm{d}x = \frac{a^x}{\ln a} + C$.

(H) $\int \cos\theta \mathrm{d}\theta = \sin\theta + C$, $\int \sin\theta \mathrm{d}\theta = -\cos\theta + C$

$\int \sec^2\theta \mathrm{d}\theta = \tan\theta + C$, $\int \csc^2\theta \mathrm{d}\theta = -\cot\theta + C$

$\int \sec\theta\tan\theta \mathrm{d}\theta = \sec\theta + C$, $\int \csc\theta\cot\theta \mathrm{d}\theta = -\csc\theta + C$

$\int \tan\theta \mathrm{d}\theta = \int \frac{\sin\theta}{\cos\theta}\mathrm{d}\theta = -\int \frac{1}{\cos\theta}\mathrm{d}(\cos\theta) = -\ln|\cos\theta| + C$

$\int \cot\theta \mathrm{d}\theta = \int \frac{\cos\theta}{\sin\theta}\mathrm{d}\theta = \int \frac{1}{\sin\theta}\mathrm{d}(\sin\theta) = \ln|\sin\theta| + C$

(I) Using double formulae $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\int \cos^2\theta d\theta = \frac{1}{2}\int (1+\cos 2\theta)d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

$$\int \sin^2\theta d\theta = \frac{1}{2}\int (1-\cos 2\theta)d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

Using triple formulae $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

i.e. $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$ and $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$

$$\int \cos^3\theta d\theta = \frac{1}{4}\int (\cos 3\theta + 3\cos\theta)d\theta = \frac{1}{12}\sin 3\theta + \frac{3}{4}\sin\theta + C$$

$$\int \sin^3\theta d\theta = \frac{1}{4}\int (3\sin\theta - \sin 3\theta)d\theta = -\frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta + C$$

Method 2

$$\int \cos^3\theta d\theta = \int \cos^2\theta\cdot\cos\theta d\theta = \int (1-\sin^2\theta)d(\sin\theta) = \sin\theta - \frac{1}{3}\sin^3\theta + C$$

$$\int \sin^3\theta d\theta = \int \sin^2\theta\cdot\sin\theta d\theta = -\int (1-\cos^2\theta)d(\cos\theta) = -\cos\theta + \frac{1}{3}\cos^3\theta + C$$

(J) Using product to sum formulae, $\int \sin m\theta\cos n\theta d\theta$ etc can be found.

e.g. $\int \sin 4\theta\cos 3\theta d\theta = \frac{1}{2}\int (\sin 7\theta + \sin\theta)d\theta = -\frac{1}{14}\cos 7\theta - \frac{1}{2}\cos\theta + C$

e.g. $\int \sin 4\theta\sin 3\theta d\theta = -\frac{1}{2}\int (\cos 7\theta - \cos\theta)d\theta = -\frac{1}{14}\sin 7\theta + \frac{1}{2}\sin\theta + C$

(K) If m is odd or n is odd, then $\int \sin^m\theta\cos^n\theta d\theta$ can be found.

e.g. $\int \sin^3\theta\cos^4\theta d\theta = -\int \sin^2\theta\cos^4\theta d(\cos\theta) = -\int (1-\cos^2\theta)\cos^4\theta d(\cos\theta)$

$= \int (-\cos^4\theta + \cos^6\theta)d(\cos\theta) = -\frac{1}{5}\cos^5\theta + \frac{1}{7}\cos^7\theta + C$

(L) If m is odd or n is even, then $\int \tan^m\theta\sec^n\theta d\theta$ can be found.

e.g. $\int \tan^3\theta\sec^5\theta d\theta = \int \tan^2\theta\sec^4\theta\cdot(\sec\theta\tan\theta)d\theta = \int (\sec^2\theta - 1)\cdot\sec^4\theta d(\sec\theta)$

$= \int (\sec^6\theta - \sec^4\theta)d(\sec\theta) = \frac{1}{7}\sec^7\theta - \frac{1}{5}\sec^5\theta + C$

e.g. $\int \tan^2\theta\sec^4\theta d\theta = \int \tan^2\theta\cdot(1+\tan^2\theta)d(\tan\theta)$

$= \frac{1}{3}\tan^3\theta + \frac{1}{4}\tan^5\theta + C$

Similarly, if m is odd or n is even, then $\int \cot^m\theta\csc^n\theta d\theta$ can be found.

e.g. $\int \cot^5\theta\csc^3\theta d\theta = \int \cot^4\theta\csc^2\theta\cdot(\csc\theta\cot\theta)d\theta = -\int (\csc^2\theta - 1)^2\cdot\csc^2\theta d(\csc\theta)$

$= \int (-\csc^6\theta + 2\csc^4\theta - \csc^2\theta)d(\csc\theta)$

$= -\frac{1}{7}\csc^7\theta + \frac{2}{5}\csc^5\theta - \frac{1}{3}\csc\theta + C$

e.g. $\int \cot^4\theta\csc^2\theta d\theta = -\int \cot^4\theta d(\cot\theta) = -\frac{1}{5}\cot^5\theta + C$

(M) $\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$

Let $y = \ln(\sec\theta + \tan\theta)$

$$\frac{dy}{d\theta} = \frac{1}{\sec\theta+\tan\theta}\cdot\left(\sec\theta\tan\theta+\sec^2\theta\right)$$

$$= \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\cdot\left(\sec\theta\right)$$

$$= \sec\theta$$

$\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C \cdots\cdots (1)$

$$J = \int \sec^3\theta d\theta = \int \sec^2\theta\sec\theta d\theta = \int \sec\theta d(\tan\theta) = \sec\theta\tan\theta - \int \tan\theta d(\sec\theta)$$

$$J = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta$$

$$= \sec\theta\tan\theta - \int \sec\theta\left(\sec^2\theta - 1\right)dx$$

$$= \sec\theta\tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$$

$2J = \sec\theta\tan\theta + \int \sec\theta d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|$ by (1)

$$J = \int \sec^3\theta d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta+\tan\theta\right| \cdots\cdots (2)$$

Let $y = \ln(\sec\theta - \tan\theta)$

$$\frac{dy}{d\theta} = \frac{1}{\sec\theta-\tan\theta}\cdot\left(\sec\theta\tan\theta-\sec^2\theta\right)$$

$$= \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\cdot\left(-\sec\theta\right)$$

$$= -\sec\theta$$

$\int \sec\theta d\theta = -\ln|\sec\theta - \tan\theta| + C \cdots\cdots (3)$

$$\sec\theta + \tan\theta = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1+\sin\theta}{\cos\theta}$$

$$= \frac{1+\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}, \text{ where } t = \tan\frac{\theta}{2}$$

$$= \frac{(1+t)^2}{(1-t)(1+t)}$$

$$= \frac{1+t}{1-t} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

$$= \frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\int \sec\theta d\theta = \ln\left|\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right| + C \cdots\cdots (4)$$

$$\sec\theta - \tan\theta = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1-\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}, \text{ where } t = \tan\frac{\theta}{2}$$

$$= \frac{(1-t)^2}{(1-t)(1+t)}$$

$$= \frac{1-t}{1+t} = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

$$= \frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}}$$

$$= \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\int \sec\theta\, d\theta = -\ln\left|\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right| + C \quad \cdots\cdots (5)$$

(N) $\int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C$

Let $y = \ln(\csc\theta + \cot\theta)$

$$\frac{dy}{d\theta} = \frac{1}{\csc\theta+\cot\theta}\cdot\left(-\csc\theta\cot\theta - \csc^2\theta\right)$$

$$= \frac{\csc\theta+\cot\theta}{\csc\theta+\cot\theta}\cdot(-\csc\theta)$$

$$= -\csc\theta$$

$$\int \csc\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + C \quad \cdots\cdots (6)$$

Let $y = \ln(\csc\theta - \cot\theta)$

$$\frac{dy}{d\theta} = \frac{1}{\csc\theta-\cot\theta}\cdot\left(-\csc\theta\cot\theta + \csc^2\theta\right)$$

$$= \frac{\csc\theta-\cot\theta}{\csc\theta-\cot\theta}\cdot(\csc\theta)$$

$$= \csc\theta$$

$$\int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C \quad \cdots\cdots (7)$$

2. **Method of substitution.**

Let $u = g(x)$ be a differentiable function and $\mathrm{f}(g(x))$ is a well defined integrable function.

$\int \mathrm{f}\left(g\left(x\right)\right)g'\left(x\right)\mathrm{d}x = \int \mathrm{f}\left(u\right)\mathrm{d}u$

Proof: Let the primitive function be $F(x)$. i.e. $\frac{\mathrm{d}F\left(u\right)}{\mathrm{d}u} = \mathrm{f}\left(u\right)$ and $\int \mathrm{f}\left(u\right)\mathrm{d}u = F\left(u\right)$

$$\frac{\mathrm{d}F\left(g\left(x\right)\right)}{\mathrm{d}x} = \frac{\mathrm{d}F\left(u\right)}{\mathrm{d}x}$$

$$= \frac{\mathrm{d}F\left(u\right)}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} \quad \text{(chain rule)}$$

$$= \mathrm{f}\left(u\right)\cdot g'\left(x\right)$$

$$= \mathrm{f}\left(g\left(x\right)\right)\cdot g'\left(x\right)$$

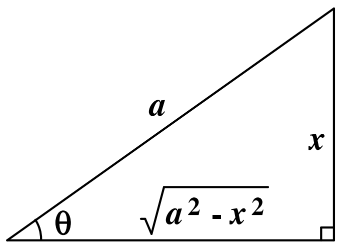
$$\mathrm{d}F\left(g\left(x\right)\right) = \mathrm{f}\left(g\left(x\right)\right)g'\left(x\right)\mathrm{d}x$$

$$\int \mathrm{d}F\left(g\left(x\right)\right) = \int \mathrm{f}\left(g\left(x\right)\right)g'\left(x\right)\mathrm{d}x$$

$$\int \mathrm{f}\left(g\left(x\right)\right)g'\left(x\right)\mathrm{d}x = F\left(g\left(x\right)\right) = \int \mathrm{f}\left(u\right)\mathrm{d}u$$

(O) $\int \frac{1}{\sqrt{a^2 - x^2}}\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + C$

Let $x = a \sin \theta$, then $\sqrt{a^2 - x^2} = a \cos \theta$, $\mathrm{d}x = a \cos \theta \, \mathrm{d}\theta$



$$\int \frac{1}{\sqrt{a^2 - x^2}}\mathrm{d}x$$

$$= \int \frac{1}{a\cos\theta}\cdot a\cos\theta \, \mathrm{d}\theta$$

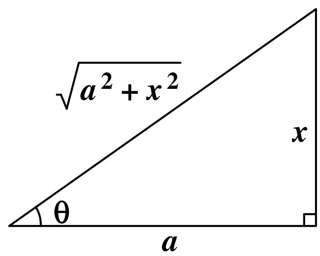
$$= \int \mathrm{d}\theta$$

$$= \theta + \mathrm{C}$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + C$$

(P) $\int \frac{1}{x^2 + a^2}\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Let $x = a \tan \theta$, then $\sqrt{a^2 + x^2} = a \sec \theta$, $\mathrm{d}x = \mathrm{a} \sec^2 \theta \, \mathrm{d}\theta$



$$\int \frac{1}{x^2 + a^2}\mathrm{d}x$$

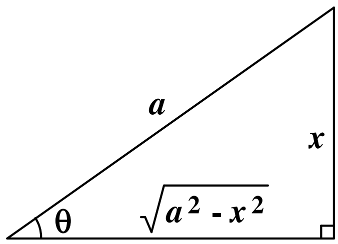
$$= \int \frac{1}{a^2\sec^2\theta}\cdot a\sec^2\theta \, \mathrm{d}\theta$$

$$= \frac{1}{a}\int \mathrm{d}\theta$$

$$= \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

(R) $\int \sqrt{a^2 - x^2}\,\mathrm{d}x = \frac{1}{2}\left(a^2 \sin^{-1}\frac{x}{a} + x\sqrt{a^2 - x^2}\right) + C$.

Let $x = a \sin \theta$, then $\sqrt{a^2 - x^2} = a \cos \theta$, $\mathrm{d}x = a \cos \theta\, \mathrm{d}\theta$



$$\int \sqrt{a^2 - x^2}\,\mathrm{d}x$$
$$= \int a^2 \cos^2 \theta \mathrm{d}\theta$$
$$= \frac{1}{2}a^2 \int (1 + \cos 2\theta)\,\mathrm{d}\theta$$
$$= \frac{1}{2}a^2\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$
$$= \frac{1}{2}a^2\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + C = \frac{1}{2}a^2(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{2}a^2\left(\sin^{-1}\frac{x}{a} + \frac{x}{a}\cdot\frac{\sqrt{a^2 - x^2}}{a}\right) + C,\ -|a| \le x \le |a|$$
$$= \frac{1}{2}\left(a^2 \sin^{-1}\frac{x}{a} + x\sqrt{a^2 - x^2}\right) + C$$