Indefinite integral (Outside syllabus)

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(3) To evaluate $\int \sqrt{x^2-a^2}\,\mathrm{d}x$.

Prerequisite $\dfrac{\mathrm{d}(\sec\theta)}{\mathrm{d}x} = \sec\theta\tan\theta$.

First, we have to evaluate $I=\int \sec\theta \mathrm{d}\theta$.

Let $y = \ln(\sec\theta + \tan\theta)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec\theta+\tan\theta}\cdot\left(\sec\theta\tan\theta+\sec^2\theta\right)$$

$$= \frac{1}{\sec\theta+\tan\theta}\cdot(\tan\theta+\sec\theta)\cdot\sec\theta = \sec\theta$$

$\therefore\ y = \int \sec\theta\mathrm{d}\theta = \ln|\sec\theta+\tan\theta| + C,\ -\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

Next, we find the integral $J=\int \sec^3\theta\mathrm{d}\theta$.

$$J = \int \sec^3\theta\mathrm{d}\theta = \int \sec\theta\cdot\sec^2\theta\mathrm{d}\theta$$

$$= \int \sec\theta\cdot\left(1+\tan^2\theta\right)\mathrm{d}\theta$$

$$= \int \sec\theta\mathrm{d}\theta + \int \sec\theta\tan^2\theta\mathrm{d}\theta$$

$$= \ln|\sec\theta+\tan\theta| + \int \tan\theta\mathrm{d}(\sec\theta)$$

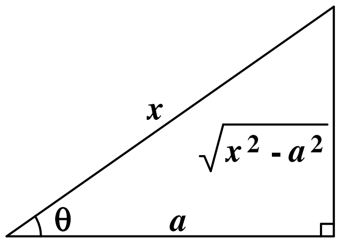
$= \ln|\sec\theta+\tan\theta| + \sec\theta\tan\theta - \int \sec\theta\mathrm{d}(\tan\theta)$, using integration by parts

$$= \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| - \int \sec^3\theta\mathrm{d}\theta$$

$2J = \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| + C_1$

$$J = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C_2$$

Let $x = a\sec\theta$, then $\sqrt{x^2-a^2} = a\tan\theta$, $\mathrm{d}x = a\,\mathrm{d}(\sec\theta)$



$$\int \sqrt{x^2-a^2}\,\mathrm{d}x$$

$$= \int a^2\tan\theta\mathrm{d}(\sec\theta)$$

$$= a^2\left(\sec\theta\tan\theta - \int \sec^3\theta\mathrm{d}\theta\right)$$

$$= a^2\left[\sec\theta\tan\theta - \left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)\right] + C_2$$

$$= \frac{1}{2}a^2\left(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right) + C_2$$

$$= \frac{1}{2}\left(x\sqrt{x^2-a^2} - a^2\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|\right) + C_2$$

$$= \frac{1}{2}\left(x\sqrt{x^2-a^2} - a^2\ln\left|x+\sqrt{x^2-a^2}\right|\right) + C,\ x \ge |a|$$

If $x \le -|a|$, let $y = -x$, then $\mathrm{d}y = -\mathrm{d}x$

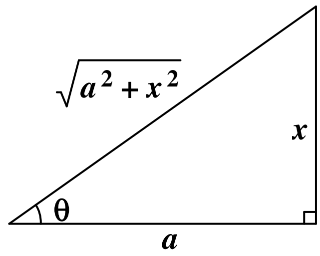
$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = -\int \sqrt{y^2-a^2}\,\mathrm{d}y = -\frac{1}{2}\left(y\sqrt{y^2-a^2} - a^2\ln\left|y+\sqrt{y^2-a^2}\right|\right) + C$$

$$= -\frac{1}{2}\left(-x\sqrt{x^2-a^2} - a^2\ln\left|-x+\sqrt{x^2-a^2}\right|\right) + C$$

$$= \frac{1}{2}\left(x\sqrt{x^2-a^2} + a^2\ln\left|-x+\sqrt{x^2-a^2}\right|\right) + C$$

(4) To evaluate $\int \sqrt{a^2+x^2}\,\mathrm{d}x$.

Let $x = a\tan\theta$, then $\sqrt{a^2+x^2} = a\sec\theta$, $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$

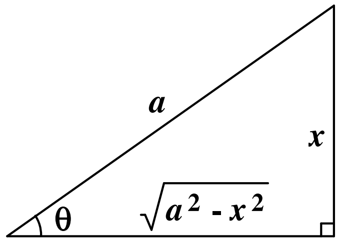


$$\int \sqrt{a^2+x^2}\,\mathrm{d}x$$
$$= \int a\sec\theta \cdot a\sec^2\theta\,\mathrm{d}\theta$$
$$= a^2\int \sec^3\theta\,\mathrm{d}\theta$$
$$= \frac{a^2}{2}\left(\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right) + C_1$$
$$= \frac{1}{2}\left(x\sqrt{a^2+x^2} + a^2\ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right|\right) + C_1$$
$$= \frac{1}{2}\left(x\sqrt{a^2+x^2} + a^2\ln\left|x+\sqrt{a^2+x^2}\right|\right) + C$$

Indefinite integral (Inside syllabus)

(1) $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$

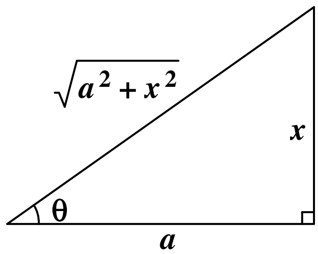
Let $x = a \sin \theta$, then $\sqrt{a^2 - x^2} = a \cos \theta$, $dx = a \cos \theta \, d\theta$



$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$$
$$= \int \frac{1}{a \cos \theta} \cdot a \cos \theta \, d\theta$$
$$= \int d\theta$$
$$= \theta + C$$
$$= \sin^{-1}\left(\frac{x}{a}\right) + C$$

(2) $\int \frac{1}{x^2 + a^2} \, dx$

Let $x = a \tan \theta$, then $\sqrt{a^2 + x^2} = a \sec \theta$, $dx = a \sec^2 \theta \, d\theta$



$$\int \frac{1}{x^2 + a^2} \, dx$$
$$= \int \frac{1}{a^2 \sec^2 \theta} \cdot a \sec^2 \theta \, d\theta$$
$$= \frac{1}{a} \int d\theta$$
$$= \frac{1}{a} \theta + C$$
$$= \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$