# Indefinite Integral Examples

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1. $\int \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}}$

Let $t = \sqrt[6]{x}$, then $x = t^6$

$\mathrm{d}x = 6t^5\,\mathrm{d}t$, $\sqrt{x} = t^3$, $\sqrt[3]{x} = t^2$

$$\int \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}} = \int \frac{6t^5\mathrm{d}t}{t^3+t^2} = 6\int \frac{t^3}{1+t}\,\mathrm{d}t$$

$$= 6\int\left(t^2 - t + 1 - \frac{1}{1+t}\right)\mathrm{d}t$$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|1+t|\right] + c$$

$$= 2t^3 - 3t^2 + 6t - 6\ln|1+t| + c$$

$= 2\sqrt{x} - 3\cdot\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left(1+\sqrt[6]{x}\right) + c$, where $c$ is a constant.

2. Breakthrough Calculus and Coordinate Geometry by Y.L. Ng & K.M. Pang
Exercise 3.8 p.103 Question 10

$$I = \int \frac{dx}{\left(x^2+b^2\right)\sqrt{x^2+a^2}} \quad (a > b > 0)$$

Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$, $x^2 + b^2 = a^2\tan^2\theta + b^2$

$$I = \int \frac{a\sec^2\theta d\theta}{\left(a^2\tan^2\theta+b^2\right)a\sec\theta}$$

$$= \int \frac{\sec\theta d\theta}{\left(a^2\tan^2\theta+b^2\right)}$$

$$= \int \frac{\cos\theta d\theta}{\left(a^2\sin^2\theta+b^2\cos^2\theta\right)}$$

$$= \int \frac{d(\sin\theta)}{\left(a^2-b^2\right)\sin^2\theta+b^2}$$

$$= \frac{1}{\sqrt{a^2-b^2}}\int \frac{d\left(\sqrt{a^2-b^2}\sin\theta\right)}{\left(a^2-b^2\right)\sin^2\theta+b^2}$$

$$= \frac{1}{\sqrt{a^2-b^2}}\int \frac{du}{u^2+b^2}, \ u = \sqrt{a^2-b^2}\sin\theta$$

$$= \frac{1}{b\sqrt{a^2-b^2}}\tan^{-1}\frac{u}{b} + C, \text{ by formula 6.4 p.82}$$

$$= \frac{1}{b\sqrt{a^2-b^2}}\cos^{-1}\frac{b}{\sqrt{u^2+b^2}} + C$$

$$= \frac{1}{b\sqrt{a^2-b^2}}\left(\frac{\pi}{2}-\sin^{-1}\frac{b}{\sqrt{\left(a^2-b^2\right)\sin^2\theta+b^2}}\right) + C, \text{ since } \sin^{-1}\alpha + \cos^{-1}\alpha = \frac{\pi}{2}$$

$$= -\frac{1}{b\sqrt{a^2-b^2}}\sin^{-1}\frac{b}{\sqrt{\frac{\left(a^2-b^2\right)x^2+b^2\left(x^2+a^2\right)}{x^2+a^2}}} + C'$$

$$= -\frac{1}{b\sqrt{a^2-b^2}}\sin^{-1}\frac{b}{a}\sqrt{\frac{x^2+a^2}{x^2+b^2}} + C'$$

3. Breakthrough Calculus and Coordinate Geometry by Y.L. Ng & K.M. Pang
Solution to Exercise 3.5 Question 16 p.95

$$\int \frac{x^3+8x-2}{\left(x^2+4x+9\right)^2}\mathrm{d}x$$

Let $\dfrac{x^3+8x-2}{\left(x^2+4x+9\right)^2} \equiv \dfrac{Ax+B}{x^2+4x+9} + \dfrac{Cx+D}{\left(x^2+4x+9\right)^2}$

$x^3 + 8x - 2 \equiv (Ax + B)(x^2 + 4x + 9) + Cx + D$

Compare coefficients,

$x^3$: $A = 1$

$x^2$: $B + 4A = 0 \Rightarrow B = -4$

$x$: $9A + 4B + C = 8 \Rightarrow C = 15$

1: $9B + D = -2 \Rightarrow D = 34$

$$\int \frac{x^3+8x-2}{\left(x^2+4x+9\right)^2}dx$$

$$= \int \frac{x-4}{x^2+4x+9}\mathrm{d}x + \int \frac{15x+34}{\left(x^2+4x+9\right)^2}\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{2x+4-12}{x^2+4x+9}\mathrm{d}x + \frac{1}{2}\int \frac{30x+60+8}{\left(x^2+4x+9\right)^2}\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{\mathrm{d}\left(x^2+4x+9\right)}{x^2+4x+9} - 6\int \frac{1}{x^2+4x+9}\mathrm{d}x + \frac{15}{2}\int \frac{\mathrm{d}\left(x^2+4x+9\right)}{\left(x^2+4x+9\right)^2} + 4\int \frac{\mathrm{d}x}{\left(x^2+4x+9\right)^2}$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - 6\int \frac{\mathrm{d}x}{(x+2)^2+5} - \frac{15}{2\left(x^2+4x+9\right)} + 4\int \frac{\mathrm{d}x}{\left[(x+2)^2+5\right]^2}$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + 4\int \frac{\mathrm{d}x}{\left[(x+2)^2+5\right]^2}$$

(Let $x + 2 = \sqrt{5}\tan\theta$)

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + 4\int \frac{\sqrt{5}\sec^2\theta\mathrm{d}\theta}{\left[\sqrt{5}\sec\theta\right]^4}$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{4\sqrt{5}}{25}\int \cos^2\theta\mathrm{d}\theta$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{2\sqrt{5}}{25}\int(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{2\sqrt{5}}{25}\left(\theta+\frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{2\sqrt{5}}{25}\left(\tan^{-1}\frac{x+2}{\sqrt{5}}+\sin\theta\cos\theta\right)$$

$$+C$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{28\sqrt{5}}{25}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{2\sqrt{5}}{25}\cdot\frac{\sqrt{5}(x+2)}{(x+2)^2+5} + C$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{28\sqrt{5}}{25}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{2}{5}\frac{(x+2)}{\left(x^2+4x+9\right)} + C$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{28\sqrt{5}}{25}\tan^{-1}\frac{x+2}{\sqrt{5}} + \frac{4x-67}{10\left(x^2+4x+9\right)} + C$$

4. To evaluate $\int \tan^6 x \sec^5 x dx$.

$\int \sec x dx = \ln|\sec x + \tan x|$ ⋯⋯ (1)

$J = \int \sec^3 x dx = \int \sec^2 x \sec x dx = \int \sec x d(\tan x) = \sec x \tan x - \int \tan x d(\sec x)$

$J = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x(\sec^2 x - 1) dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$

$2J = \sec x \tan x + \int \sec x dx = \sec x \tan x + \ln|\sec x + \tan x|$ by (1)

$J = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$ ⋯⋯ (2)

$\frac{d}{dx}\left((m+n-3)\tan^{m-1} x \sec^n x - (m-1)\tan^{m-1} x \sec^{n-2} x\right)$

$=(m+n-3)(m-1)\tan^{m-2} x \sec^{n+2} x + n(m+n-3)\tan^m x \sec^n x - (m-1)^2\tan^{m-2} x \sec^n x - (m-1)(n-2)\tan^m x \sec^{n-2} x$

$= (m+n-3)\tan^{m-2} x \sec^n x[(m-1)\sec^2 x + n\tan^2 x] - (m-1)\tan^{m-2} x \sec^{n-2} x[(m-1)\sec^2 x + (n-2)\tan^2 x]$

$= (m+n-3)\tan^{m-2} x \sec^n x[(m-1)(1+\tan^2 x) + n\tan^2 x] - (m-1)\tan^{m-2} x \sec^{n-2} x[(m-1)\sec^2 x + (n-2)(\sec^2 x - 1)]$

$= (m+n-3)\tan^{m-2} x \sec^n x[(m+n-1)\tan^2 x + (m-1)] - (m-1)\tan^{m-2} x \sec^{n-2} x[(m+n-3)\sec^2 x - (n-2)]$

$= (m+n-3)(m+n-1)\tan^m x \sec^n x + (m+n-3)(m-1)\tan^{m-2} x \sec^n x - (m-1)(m+n-3)\tan^{m-2} x \sec^n x + (m-1)(n-2)\tan^{m-2} x \sec^{n-2} x$

$= (m+n-3)(m+n-1)\tan^m x \sec^n x + (n-2)(m-1)\tan^{m-2} x \sec^{n-2} x$

$(m+n-3)\tan^{m-1} x \sec^n x + (m-1)\tan^{m-1} x \sec^{n-2} x$

$= (m+n-3)(m+n-1)\int \tan^m x \sec^n x dx + (m-1)(n-2)\int \tan^{m-2} x \sec^{n-2} x dx$

If $I_{m,n} = \int \tan^m x \sec^n x dx$, then

$$I_{m,n} = \frac{\tan^{m-1} x \sec^n x}{m+n-1} + \frac{(m-1)\tan^{m-1} x \sec^{n-2} x}{(m+n-3)(m+n-1)} - \frac{(m-1)(n-2)}{(m+n-3)(m+n-1)} I_{m-2,n-2}$$

$I_{2,1} = \int \tan^2 x \sec x dx = \int \tan x d(\sec x) = \sec x \tan x - \int \sec^3 x dx = \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x|$ by (2)

$$I_{4,3} = \frac{\tan^3 x \sec^3 x}{6} + \frac{3\tan^3 x \sec x}{4\times 6} - \frac{3}{4\times 6} I_{2,1} = \frac{\tan^3 x \sec^3 x}{6} + \frac{\tan^3 x \sec x}{8} - \frac{1}{8} I_{2,1}$$

$$= \frac{\tan^3 x \sec^3 x}{6} + \frac{\tan^3 x \sec x}{8} - \frac{1}{8}\left(\frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x|\right)$$

$$= \frac{\tan^3 x \sec^3 x}{6} + \frac{\tan^3 x \sec x}{8} - \frac{\sec x \tan x}{16} + \frac{1}{16}\ln|\sec x + \tan x|$$

$$I_{6,5} = \frac{\tan^5 x \sec^5 x}{10} + \frac{5\tan^5 x \sec^3 x}{8\times 10} - \frac{5\times 3}{8\times 10} I_{4,3} = \frac{\tan^5 x \sec^5 x}{10} + \frac{\tan^5 x \sec^3 x}{16} - \frac{3}{16} I_{4,3}$$

$$= \frac{\tan^5 x \sec^5 x}{10} + \frac{\tan^5 x \sec^3 x}{16} - \frac{3}{16}\left(\frac{\tan^3 x \sec^3 x}{6} + \frac{\tan^3 x \sec x}{8} - \frac{\sec x \tan x}{16} + \frac{1}{16}\ln|\sec x + \tan x|\right)$$

$$= \frac{\tan^5 x \sec^5 x}{10} + \frac{\tan^5 x \sec^3 x}{16} - \frac{\tan^3 x \sec^3 x}{32} - \frac{3\tan^3 x \sec x}{128} + \frac{3\sec x \tan x}{256} - \frac{3}{256}\ln|\sec x + \tan x|$$

5. Calculus and Analytic Geometry II by K.S. Ng & Y.K. Kwok P.42 Q5

Evaluate $\int \frac{\mathrm{d}x}{x+\sqrt{x^2-1}}$ by using the substitution $u = x+\sqrt{x^2-1}$.

$(u - x)^2 = x^2 - 1$

$u^2 - 2xu + x^2 = x^2 - 1$

$u^2 + 1 = 2xu$

$x = \frac{u^2+1}{2u}$

$\mathrm{d}x = \frac{u\cdot(2u)-(u^2+1)}{2u^2}\mathrm{d}u = \frac{u^2-1}{2u^2}\mathrm{d}u$

Let $I = \int \frac{\mathrm{d}x}{x+\sqrt{x^2-1}}$

$$\begin{aligned}
I &= \frac{1}{2}\int \frac{1}{u}\cdot\frac{u^2-1}{u^2}\mathrm{d}u \\
&= \frac{1}{2}\int\left(\frac{1}{u}-\frac{1}{u^3}\right)\mathrm{d}u \\
&= \frac{1}{2}\left(\ln|u|+\frac{1}{2u^2}\right)+c \\
&= \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+\frac{1}{4\left(x+\sqrt{x^2-1}\right)^2}+c \\
&= \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+\frac{1}{4\left(x^2+2x\sqrt{x^2-1}+x^2-1\right)}+c \\
&= \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+\frac{1}{4\left(2x^2-1+2x\sqrt{x^2-1}\right)}+c \\
&= \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+\frac{1}{4\left(2x^2-1+2x\sqrt{x^2-1}\right)}\cdot\frac{\left(2x^2-1-2x\sqrt{x^2-1}\right)}{\left(2x^2-1-2x\sqrt{x^2-1}\right)}+c \\
&= \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+\frac{2x^2-1-2x\sqrt{x^2-1}}{4\left[4x^4-4x^2+1-4x^2\left(x^2-1\right)\right]}+c \\
&= \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+\frac{2x^2-1-2x\sqrt{x^2-1}}{4}+c \\
&= \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+\frac{x^2}{2}-\frac{x\sqrt{x^2-1}}{2}+c
\end{aligned}$$

**Method 2**

$$\int \frac{\mathrm{d}x}{x+\sqrt{x^2-1}} = \int \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\,\mathrm{d}x$$

$$= \int \frac{x-\sqrt{x^2-1}}{x^2-\left(x^2-1\right)}\,\mathrm{d}x$$

$$= \int \left(x-\sqrt{x^2-1}\right)\mathrm{d}x$$

$$= \frac{1}{2}x^2 - \int \tan\theta\left(\sec\theta\tan\theta\right)\mathrm{d}\theta\text{, } x = \sec\theta,\ \sqrt{x^2-1} = \tan\theta,\ \mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$$

$$= \frac{1}{2}x^2 - \int \left(\sec^3\theta - \sec\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{2}x^2 - \int \sec^3\theta\,\mathrm{d}\theta + \ln\left|\sec\theta + \tan\theta\right| + C$$

Let $J = \int \sec^3\theta\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}(\tan\theta) = \sec\theta\tan\theta - \int \tan\theta\,\mathrm{d}(\sec\theta)$

$$= x\sqrt{x^2-1} - \int \sec\theta\tan^2\theta\,\mathrm{d}\theta = x\sqrt{x^2-1} - J + \int \sec\theta\,\mathrm{d}\theta$$

$$2J = x\sqrt{x^2-1} + \ln\left|\sec\theta + \tan\theta\right| + C$$

$$J = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C$$

$$\int \frac{\mathrm{d}x}{x+\sqrt{x^2-1}} = \frac{1}{2}x^2 - J + \ln\left|x+\sqrt{x^2-1}\right| + C$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + \ln\left|x+\sqrt{x^2-1}\right| + C$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C$$

6. Calculus and Analytic Geometry II by K.S. Ng & Y.K. Kwok P.42 Q6

Evaluate $\int \frac{\mathrm{d}x}{x-\sqrt{x^2+1}}$ by using the substitution $u=\frac{\sqrt{x^2+1}-1}{x}$.

$ux+1=\sqrt{x^2+1}$ (Note that $u=\frac{\sqrt{x^2+1}-1}{x}\le\frac{x+1-1}{x}=1$ for $x>0$)

$u^2x^2+2ux+1=x^2+1$

$(u^2-1)x^2+2ux=0$

$$x=\frac{2u}{1-u^2};\ \mathrm{d}x=2\cdot\frac{(1-u^2)-u(-2u)}{(1-u^2)^2}\cdot\mathrm{d}u=\frac{2(1+u^2)}{(1-u^2)^2}\mathrm{d}u$$

$$x-\sqrt{x^2+1}=\frac{2u}{1-u^2}-\sqrt{\left(\frac{2u}{1-u^2}\right)^2+1}=\frac{2u}{1-u^2}-\sqrt{\frac{4u^2+1-2u^2+u^4}{(1-u^2)^2}}=\frac{2u-\sqrt{1+2u^2+u^4}}{1-u^2}=\frac{2u-(1+u^2)}{1-u^2}$$

$$=-\frac{(1-u)^2}{(1-u)(1+u)}=-\frac{1-u}{1+u}=\frac{u-1}{u+1}$$

$$\int\frac{\mathrm{d}x}{x-\sqrt{x^2+1}}=\int\frac{u+1}{u-1}\cdot\frac{2(1+u^2)}{(1-u^2)^2}\mathrm{d}u=-\int\frac{u+1}{(1-u)}\cdot\frac{2(1+u^2)}{(1+u)^2(1-u)^2}\mathrm{d}u=-2\int\frac{1+u^2}{(1-u)^3(1+u)}\mathrm{d}u$$

Let $\frac{1+u^2}{(1-u)^3(1+u)}\equiv\frac{A}{1-u}+\frac{B}{(1-u)^2}+\frac{C}{(1-u)^3}+\frac{D}{1+u}$

$1+u^2\equiv A(1-u)^2(1+u)+B(1-u)(1+u)+C(1+u)+D(1-u)^3$

Put $u=-1$: $D=\frac{1}{4}$

Put $u=1$: $C=1$

Differentiate and put $u=1$, $2=-2B+C\Rightarrow B=-\frac{1}{2}$

Compare coefficients of $u^3$: $A-D=0\Rightarrow A=\frac{1}{4}$

$$\int\frac{\mathrm{d}x}{x-\sqrt{x^2+1}}=-2\int\left(\frac{1}{4}\cdot\frac{1}{1-u}-\frac{1}{2(1-u)^2}+\frac{1}{(1-u)^3}+\frac{1}{4}\cdot\frac{1}{1+u}\right)\mathrm{d}u$$

$$=-2\left[-\frac{1}{4}\ln|1-u|-\frac{1}{2(1-u)}+\frac{1}{2(1-u)^2}+\frac{1}{4}\ln|1+u|\right]+C$$

$$=\frac{1}{1-u}-\frac{1}{(1-u)^2}+\frac{1}{2}\ln\left|\frac{1-u}{1+u}\right|+C$$

$$1-u=1-\frac{\sqrt{x^2+1}-1}{x}=\frac{x+1-\sqrt{x^2+1}}{x};\ 1+u=\frac{x-1+\sqrt{x^2+1}}{x}$$

$$\frac{1-u}{1+u}=\frac{x+1-\sqrt{x^2+1}}{x-1+\sqrt{x^2+1}}=\frac{(x+1-\sqrt{x^2+1})(x-1-\sqrt{x^2+1})}{-2x}=\frac{(x-\sqrt{x^2+1})^2-1}{-2x}$$

$$=\frac{x^2-2x\sqrt{x^2+1}+x^2+1-1}{-2x}=\frac{2x^2-2x\sqrt{x^2+1}}{-2x}=\sqrt{x^2+1}-x$$

$$\frac{1}{1-u}=\frac{x}{(x+1-\sqrt{x^2+1})}\cdot\frac{(x+1+\sqrt{x^2+1})}{(x+1+\sqrt{x^2+1})}=\frac{x(x+1+\sqrt{x^2+1})}{2x}=\frac{x+1+\sqrt{x^2+1}}{2}$$

$$-\frac{1}{(1-u)^2}=-\frac{x^2}{\left(x+1-\sqrt{x^2+1}\right)^2}\cdot\frac{\left(x+1+\sqrt{x^2+1}\right)^2}{\left(x+1+\sqrt{x^2+1}\right)^2}=-\frac{x^2\left[2x^2+2x+2+2(x+1)\sqrt{x^2+1}\right]}{(2x)^2}$$

$$=-\frac{x^2+x+1+(x+1)\sqrt{x^2+1}}{2}$$

$$\int\frac{\mathrm{d}x}{x-\sqrt{x^2+1}}=\frac{x+1+\sqrt{x^2+1}}{2}-\frac{x^2+x+1+(x+1)\sqrt{x^2+1}}{2}+\frac{1}{2}\ln\left|\sqrt{x^2+1}-x\right|+C$$

$$=\frac{1}{2}\ln\left|\sqrt{x^2+1}-x\right|-\frac{x^2}{2}-\frac{1}{2}x\sqrt{x^2+1}+C$$

**Method 2**

$$\int\frac{\mathrm{d}x}{x-\sqrt{x^2+1}}=\int\frac{1}{x-\sqrt{x^2+1}}\cdot\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}\,\mathrm{d}x$$

$$=\int\frac{x+\sqrt{x^2+1}}{x^2-\left(x^2+1\right)}\,\mathrm{d}x$$

$$=-\int\left(x+\sqrt{x^2+1}\right)\mathrm{d}x$$

$$=-\frac{x^2}{2}-\int\sqrt{x^2+1}\,\mathrm{d}x$$

$$=-\frac{x^2}{2}-\int\sqrt{\tan^2\theta+1}\sec^2\theta\,\mathrm{d}\theta,\ x=\tan\theta,\ \mathrm{d}x=\sec^2\theta\,\mathrm{d}\theta$$

$$=-\frac{x^2}{2}-\int\sec^3\theta\,\mathrm{d}\theta$$

$$=-\frac{x^2}{2}-\frac{1}{2}\sec\theta\tan\theta-\frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|+C$$

$$=-\frac{x^2}{2}-\frac{1}{2}\cdot x\sqrt{1+x^2}-\frac{1}{2}\ln\left|\sqrt{1+x^2}+x\right|+C$$

7. Calculus and Analytic Geometry II by K.S. Ng & Y.K. Kwok P.55 Q21

Let $m$, $n$ be non-negative integers and $I_{m,n} = \int \cos^m x \sin nx \, dx$.

Prove that for $m \ge 1$, $n \ge 1$, $(m + n)I_{m,n} = mI_{m-1,n-1} - \cos^m x \cos nx$.

$$\begin{aligned}
(m+n)I_{m,n} - mI_{m-1,n-1} &= m\int\left[\cos^m x\sin nx - \cos^{m-1} x\sin(n-1)x\right]dx + n\int\cos^m x\sin nx\,dx \\
&= m\int\cos^{m-1} x\left[\cos x\sin nx - \sin(n-1)x\right]dx + n\int\cos^m x\sin nx\,dx \\
&= m\int\cos^{m-1} x\cdot\frac{1}{2}\left[\sin(n+1)x + \sin(n-1)x - 2\sin(n-1)x\right]dx + nI_{m,n} \\
&= \frac{m}{2}\int\cos^{m-1} x\left[\sin(n+1)x - \sin(n-1)x\right]dx + nI_{m,n} \\
&= m\int\cos^{m-1} x\left(\cos nx\sin x\right)dx + nI_{m,n} \quad \text{...... (1)}
\end{aligned}$$

Let $y = -\cos^m x \cos nx$

$\frac{dy}{dx} = n \cos^m x \sin nx + m \cos^{m-1} x \cos nx \sin x$

$y = n\, I_{m,n} + m\int\cos^{m-1} x\left(\cos nx\sin x\right)dx$

$-\cos^m x \cos nx = n\, I_{m,n} + (m + n)I_{m,n} - mI_{m-1,n-1} - n\, I_{m,n}$ by (1)

$\therefore$ For $m \ge 1$, $n \ge 1$, $(m + n)I_{m,n} = mI_{m-1,n-1} - \cos^m x \cos nx$

8. $\mathrm{I}=\int \frac{\mathrm{d}x}{\left(4x^2+64\right)^{\frac{3}{2}}}=\int \frac{\mathrm{d}x}{8\left(x^2+16\right)^{\frac{3}{2}}}$

Let $x = 4 \tan \theta$, then $\sqrt{x^2+16} = 4 \sec \theta$, $\mathrm{d}x = 4 \sec^2 \theta \, \mathrm{d}\theta$

$\mathrm{I}=\int \frac{4\sec^2\theta\mathrm{d}\theta}{8\left(4\sec\theta\right)^3}=\frac{1}{128}\int\cos\theta\mathrm{d}\theta=\frac{1}{128}\sin\theta+C=\frac{1}{128}\cdot\frac{x}{\sqrt{x^2+16}}+C$

Check

Let $y=\frac{1}{128}\cdot\frac{x}{\sqrt{x^2+16}}+C$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{128}\cdot\frac{\sqrt{x^2+16}-\frac{x}{2\sqrt{x^2+16}}\cdot 2x}{x^2+16}$$

$$=\frac{1}{128}\cdot\frac{x^2+16-x^2}{\left(x^2+16\right)^{\frac{3}{2}}}$$

$$=\frac{1}{8\left(x^2+16\right)^{\frac{3}{2}}}$$

$$=\frac{1}{\left(4x^2+64\right)^{\frac{3}{2}}}$$