Fundamental Theorem of Calculus

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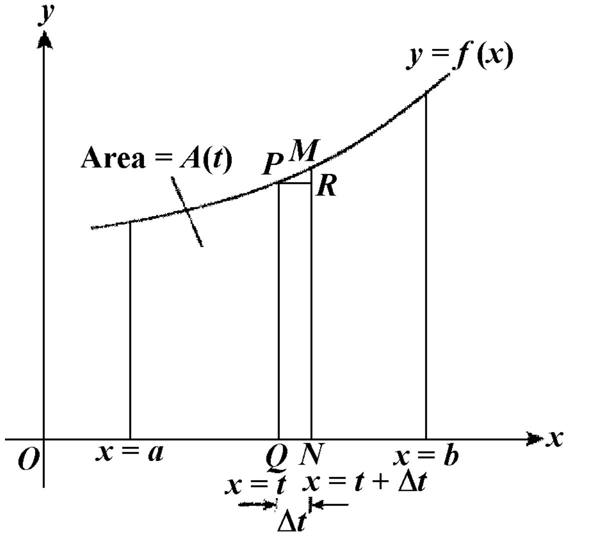
If $\int f(x)\,dx = F(x) + C$, i.e. $\frac{d}{dx}F(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

and $F(b) - F(a)$ is usually denoted by $\left[F(x)\right]_a^b$ or $\left[\int f(x)\,dx\right]_a^b$.

Proof: In the figure, $y = f(x)$ is a continuous curve in the interval $[a, b]$.



Suppose t and $t + \Delta t$ are in $[a, b]$.

Let $A(t) = \int_a^t f(x)\,dx$, i.e. the area bounded by $y = f(x)$, $x = a$, $x = t$ and the x-axis.

$\therefore A(t + \Delta t) = \int_a^{t+\Delta t} f(x)\,dx$

Consider $A(t + \Delta t) - A(t) = \int_a^{t+\Delta t} f(x)\,dx - \int_a^t f(x)\,dx$

$= \int_t^{t+\Delta t} f(x)\,dx$

When $\Delta t \to 0$, the area of the rectangle $PQNR$ approaches that of $PQNM$, i.e. $\int_t^{t+\Delta t} f(x)\,dx \to f(t)\Delta t$

$$\therefore \lim_{\Delta t\to 0}\frac{A(t+\Delta t)-A(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\int_t^{t+\Delta t} f(x)\,dx}{\Delta t} = \lim_{\Delta t\to 0}\frac{f(t)\Delta t}{\Delta t} = f(t)$$

i.e. $\frac{d}{dt}A(t) = f(t)$

Thus $A(t)$ is a primitive function of $f(t)$ and differs from $F(t)$, another primitive function of $f(t)$ $\left(\text{i.e. } \frac{d}{dt}F(t) = f(t)\right)$, by an arbitrary constant C .

Hence $A(t) = F(t) + C$.

$\therefore \int_a^t f(x)\,dx = F(t) + C \cdots\cdots (*)$

Putting $t = a$ into (*), $\int_a^a f(x)\,dx = F(a) + C$

$0 = F(a) + C$

$C = -F(a)$

Putting $t = b$ into (*), we have $\int_a^b f(x)\,dx = F(b) - F(a)$