Examples on Periodic Function

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Example 1

$y = |\sin x|$ is periodic with period = π

$G(x) = \int_0^x |\sin t|\,dt$, Show that $G(x)$ is not a periodic function.

If $0 \le x \le \pi$, then for all t such that $0 \le t \le x$, $|\sin t| = \sin t$.

$G(x) = \int_0^x |\sin t|\,dt = \int_0^x \sin t\,dt = -\cos x + \cos 0 = 1 - \cos x$

$G(\pi) = \int_0^\pi |\sin t|\,dt = 1 - \cos \pi = 2$

$G(2\pi) = \int_0^\pi |\sin t|\,dt + \int_\pi^{2\pi} |\sin t|\,dt = 2 + 2 = 4$

$G(3\pi) = 6$, $G(4\pi) = 8$, $\cdots$, $G(n\pi) = 2n$ where n is any integer.

If $G(x)$ is a periodic function, then there is a positive constant T such that $G(x + T) = G(x)$

If $T > \pi$, let m be the greatest positive integer such that $T = m\pi + a$, where $0 \le a < \pi$

Let $x = \pi - a$, $G(x + T) = G(x) \Rightarrow G(\pi - a + m\pi + a) = G(\pi - a)$

$G((m + 1)\pi) = G(\pi - a)$

$2(m + 1) = 1 - \cos(\pi - a)$

$2m + 2 = 1 + \cos a$

$2m + 1 = \cos a$

contradict to the fact that $-1 \le \cos a \le 1$

If the period $T < \pi$

However, let $x = \pi - T$, $G(x + T) = G(x) \Rightarrow G(\pi - T + T) = G(\pi - T)$

$G(\pi) = G(\pi - T)$

$2 = 1 - \cos T$

$\cos T = -1$

$T = \pi$

contradict to the fact that $T < \pi$

If $T = \pi$, let $x = \pi$, then $G(x + T) = G(x)$

$\Rightarrow G(\pi + \pi) = G(\pi) \Rightarrow 4 = 2$, contradiction

Therefore G cannot be a periodic function.

Example 2 (Advanced Level Pure Mathematics Calculus and Analytical Geometry II by K.S.Ng, Y.K. Kwok p.86 Q7)

(a) If f is a continuous periodic function of period $2c$, show that $\int_{-c}^{c} \mathrm{f}(x)\,\mathrm{d}x = \int_{-c+a}^{c+a} \mathrm{f}(x)\,\mathrm{d}x$, where a is a real constant.

(b) If f is an even periodic function of period 2π, evaluate the definite integral $\int_0^{2\pi} \mathrm{f}(x)\sin x\,\mathrm{d}x$.

(a) $\int_{-c+a}^{c+a} \mathrm{f}(x)\,\mathrm{d}x = \int_{-c+a}^{c} \mathrm{f}(x)\,\mathrm{d}x + \int_{c}^{c+a} \mathrm{f}(x)\,\mathrm{d}x$, for the 2nd integral, let $u = x - 2c$

$$= \int_{-c+a}^{c} \mathrm{f}(x)\,\mathrm{d}x + \int_{-c}^{-c+a} \mathrm{f}(u+2c)\,\mathrm{d}u = \int_{-c+a}^{c} \mathrm{f}(x)\,\mathrm{d}x + \int_{-c}^{-c+a} \mathrm{f}(u)\,\mathrm{d}u$$

$$= \int_{-c+a}^{c} \mathrm{f}(x)\,\mathrm{d}x + \int_{-c}^{-c+a} \mathrm{f}(x)\,\mathrm{d}x = \int_{-c}^{c} \mathrm{f}(x)\,\mathrm{d}x$$

(b) $\int_0^{2\pi} \mathrm{f}(x)\sin x\,\mathrm{d}x = \int_0^{\pi} \mathrm{f}(x)\sin x\,\mathrm{d}x + \int_{\pi}^{2\pi} \mathrm{f}(x)\sin x\,\mathrm{d}x$, for the 2nd integral, let $v = 2\pi - x$

$$= \int_0^{\pi} \mathrm{f}(x)\sin x\,\mathrm{d}x + \int_{\pi}^{0} \mathrm{f}(2\pi - v)\sin(2\pi - v)(-\mathrm{d}v)$$

$$= \int_0^{\pi} \mathrm{f}(x)\sin x\,\mathrm{d}x + \int_0^{\pi} \mathrm{f}(-v)(-\sin v)\,\mathrm{d}v \quad \because \text{ f is periodic with period } 2\pi$$

$$= \int_0^{\pi} \mathrm{f}(x)\sin x\,\mathrm{d}x - \int_0^{\pi} \mathrm{f}(v)\sin v\,\mathrm{d}v \quad \because \text{ f is even}$$

$$= 0$$