$$\int_0^{\frac{\pi}{4}} \frac{1}{1+\tan x}\,dx$$

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$I=\int_0^{\frac{\pi}{4}} \frac{1}{1+\tan x}\,dx$. Let $t=\tan\frac{x}{2}$, then $dt=\frac{1}{2}\sec^2\frac{x}{2}dx=\frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx \Rightarrow dx=\frac{2dt}{1+t^2}$

When $x = 0$, $t = 0$; when $x=\frac{\pi}{4}$, $t=\tan\frac{\pi}{8}=-1+\sqrt{2}$

$$\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}=\frac{2t}{1-t^2}$$

$$I=\int_0^{-1+\sqrt{2}} \frac{1}{1+\frac{2t}{1-t^2}}\cdot\frac{2dt}{(1+t^2)}$$

$$=2\int_0^{-1+\sqrt{2}} \frac{1-t^2}{1-t^2+2t}\cdot\frac{dt}{(1+t^2)}$$

Let $\frac{1-t^2}{1-t^2+2t}\cdot\frac{1}{(1+t^2)}\equiv\frac{At+B}{1-t^2+2t}+\frac{Ct+D}{1+t^2}$

$1 - t^2 \equiv (At + B)(1 + t^2) + (Ct + D)(1 - t^2 + 2t)$ ⋯⋯ (*)

Put $t = i = \sqrt{-1}$: $1 + 1 = 0 + (Ci + D)(1 + 1 + 2i)$

$2 = (Ci + D)(2 + 2i)$

$1 = (D - C) + (C + D)i$

Compare the real parts: $D - C = 1$ ⋯⋯ (1)

Compare imaginary parts: $C + D = 0 \Rightarrow D = -C$ ⋯⋯ (2)

Sub. (2) into (1): $-2C = 1 \Rightarrow C=-\frac{1}{2}$, $D=\frac{1}{2}$

Compare coefficients of x^3 int (*): $0 = A - C \Rightarrow A = C=-\frac{1}{2}$

Compare the constant term in (*): $1 = B + D \Rightarrow B =1-\frac{1}{2}=\frac{1}{2}$

$$I=2\int_0^{-1+\sqrt{2}}\left(\frac{-\frac{1}{2}t+\frac{1}{2}}{1-t^2+2t}+\frac{-\frac{1}{2}t+\frac{1}{2}}{1+t^2}\right)dt=\int_0^{-1+\sqrt{2}}\left(\frac{t-1}{t^2-2t-1}-\frac{t-1}{t^2+1}\right)dt$$

$$=\int_0^{-1+\sqrt{2}}\frac{t-1}{t^2-2t-1}dt-\int_0^{-1+\sqrt{2}}\frac{t-1}{t^2+1}dt=\int_0^{-1+\sqrt{2}}\frac{2t-2}{2(t^2-2t-1)}dt-\int_0^{-1+\sqrt{2}}\frac{2t-2}{2(t^2+1)}dt$$

$$=\frac{1}{2}\int_0^{-1+\sqrt{2}}\frac{d(t^2-2t-1)}{(t^2-2t-1)}-\frac{1}{2}\int_0^{-1+\sqrt{2}}\frac{d(t^2+1)}{(t^2+1)}+\frac{2}{2}\int_0^{-1+\sqrt{2}}\frac{dt}{t^2+1}$$

$$=\left(\frac{1}{2}\ln\left|t^2-2t-1\right|-\frac{1}{2}\ln\left|t^2+1\right|+\tan^{-1}t\right)_0^{-1+\sqrt{2}}$$

$$=\frac{1}{2}\left(\ln\left|(-1+\sqrt{2})^2-2(-1+\sqrt{2})-1\right|-\ln\left|(-1+\sqrt{2})^2+1\right|\right)+\tan^{-1}(-1+\sqrt{2})$$

$$=\frac{1}{2}\left(\ln\left|3-2\sqrt{2}+2-2\sqrt{2}-1\right|-\ln\left|4-2\sqrt{2}\right|\right)+\frac{\pi}{8}=\frac{1}{2}\left(\ln\left|4-4\sqrt{2}\right|-\ln\left|4-2\sqrt{2}\right|\right)+\frac{\pi}{8}$$

$$=\frac{1}{2}\ln\frac{4(\sqrt{2}-1)}{2(2-\sqrt{2})}+\frac{\pi}{8}=\frac{1}{2}\ln\frac{4(\sqrt{2}-1)}{2\sqrt{2}(\sqrt{2}-1)}+\frac{\pi}{8}=\frac{1}{2}\ln\frac{2}{\sqrt{2}}+\frac{\pi}{8}=\frac{1}{2}\ln\sqrt{2}+\frac{\pi}{8}=\frac{1}{4}\ln 2+\frac{\pi}{8}$$

Method 2

$$I=\int_0^{\frac{\pi}{4}}\frac{1}{1+\tan x}\mathrm{d}x=\int_0^{\frac{\pi}{4}}\frac{\cos x}{\sin x+\cos x}\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{4}}\frac{\cos x}{\sin x+\cos x}\cdot\frac{\cos x-\sin x}{\cos x-\sin x}\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{4}}\frac{\cos^2 x-\sin x\cos x}{\cos^2 x-\sin^2 x}\mathrm{d}x$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\frac{1}{2}(1+\cos 2x)-\frac{1}{2}\sin 2x}{\cos 2x}\mathrm{d}(2x)$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{1+\cos u-\sin u}{\cos u}\mathrm{d}u$$, $u = 2x$, when $x = 0$, $u = 0$; when $x=\frac{\pi}{4}$, $u=\frac{\pi}{2}$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}\left(\sec u+1-\frac{\sin u}{\cos u}\right)\mathrm{d}u$$

$$=\frac{1}{4}\left(\ln\left|\sec u+\tan u\right|+u+\ln\left|\cos u\right|\right)_0^{\frac{\pi}{2}}$$

$$=\frac{1}{4}\left(\ln\left|1+\sin u\right|+u\right)_0^{\frac{\pi}{2}}$$

$$=\frac{1}{4}\left(\ln\left|1+1\right|+\frac{\pi}{2}\right)=\frac{1}{4}\ln 2+\frac{\pi}{8}$$