Problem on Integration

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This is a hard problem on integration. It involves the use of substitutions.

Consider the following improper integral: $\int_a^b \left[(x-a)(b-x)^2\right]^{-\frac{1}{3}} \mathrm{d}x$, $a < b$.

(a) By using substitution $x = a\cos^2\theta + b\sin^2\theta$, prove that is equal to $2\int_0^{\frac{\pi}{2}} \tan^{\frac{1}{3}}\theta \mathrm{d}\theta$.

(b) By using the substitution $t = \tan^{\frac{1}{3}}\theta$, prove that the integral is equal to $6\int_0^{\infty} \frac{t^3}{1+t^6}\mathrm{d}t$,

hence prove that is also equal to $3\int_0^{\infty} \frac{u}{1+u^3}\mathrm{d}u$.

(c) Prove that $\int_0^{\infty} \frac{v}{1+v^3}\mathrm{d}v = \int_0^{\infty} \frac{1}{1+v^3}\mathrm{d}v$. Hence find the integral in part (a).

(d) Use the substitution $x = \frac{a+bu}{1+u}$ to do part (a) again.

Solution

(a) $x = a\cos^2\theta + b\sin^2\theta$; $x = a$, $\theta = 0$; $x = b$, $\theta = \frac{\pi}{2}$.

$x - a = (b-a)\sin^2\theta$; $b - x = (b-a)\cos^2\theta$; $\mathrm{d}x = 2(b-a)\sin\theta\cos\theta\,\mathrm{d}\theta$

$$I = \int_0^{\frac{\pi}{2}} \left[(b-a)\sin^2\theta(b-a)^2\cos^4\theta\right]^{-\frac{1}{3}} 2(b-a)\sin\theta\cos\theta \mathrm{d}\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \tan^{\frac{1}{3}}\theta \mathrm{d}\theta$$

(b) $t = \tan^{\frac{1}{3}}\theta$; $\theta = 0$, $t = 0$, $\theta \to \frac{\pi}{2}$, $t \to \infty$

$\mathrm{d}t = \frac{1}{3}\tan^{-\frac{2}{3}}\theta\sec^2\theta\,\mathrm{d}\theta \Rightarrow \mathrm{d}\theta = \frac{3t^2\mathrm{d}t}{1+t^6}$

$$I = 2\int_0^{\infty} t\cdot\frac{3t^2\mathrm{d}t}{1+t^6} = 6\int_0^{\infty}\frac{t^3}{1+t^6}\mathrm{d}t$$

Let $u = t^2$; $t = 0$, $u = 0$; $t \to \infty$, $u \to \infty$; $\mathrm{d}u = 2t\,\mathrm{d}t$

$$I = 3\int_0^{\infty}\frac{t^2}{1+(t^2)^3}\cdot 2t\mathrm{d}t = 3\int_0^{\infty}\frac{u}{1+u^3}\mathrm{d}u$$

(c) $\text{J} = \int_0^{\infty} \frac{v}{1+v^3}\,\text{d}v$. Let $w = \frac{1}{v}$; $v \to 0,\ w \to \infty$; $v \to \infty,\ w \to 0$; $\text{d}v = -\frac{1}{w^2}\text{d}w$

$$\text{J} = \int_{\infty}^{0} \frac{\frac{1}{w}}{1+\left(\frac{1}{w}\right)^3}\left(-\frac{1}{w^2}\right)\text{d}w.$$

$$\text{J} = \int_0^{\infty} \frac{1}{1+w^3}\,\text{d}w = \int_0^{\infty} \frac{1}{1+v^3}\,\text{d}v$$

$$\text{I} = 3\text{J} = 3\int_0^{\infty} \frac{1}{1+v^3}\,\text{d}v$$

$$\text{I} + \text{I} = 3\left(\int_0^{\infty} \frac{1}{1+v^3}\,\text{d}v + \int_0^{\infty} \frac{v}{1+v^3}\,\text{d}v\right)$$

$$2\text{I} = 3\int_0^{\infty} \frac{1+v}{1+v^3}\,\text{d}v$$

$$= 3\int_0^{\infty} \frac{1}{1-v+v^2}\,\text{d}v$$

$$= 3\int_0^{\infty} \frac{1}{\left(v-\frac{1}{2}\right)^2+\frac{3}{4}}\,\text{d}v$$

$$= 3\int_0^{\infty} \frac{1}{\frac{3}{4}\left[\left(\frac{v-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1\right]}\,\text{d}\left(v-\frac{1}{2}\right)$$

$$= 2\sqrt{3}\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right)\Bigg|_0^{\infty}$$

$$= 2\sqrt{3}\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = \frac{4\sqrt{3}\pi}{3}$$

$$\int_a^b \left[(x-a)(b-x)^2\right]^{-\frac{1}{3}}\text{d}x = \text{I} = \frac{2\sqrt{3}\pi}{3}$$

(d) $x = \dfrac{a+bu}{1+u}$; $x = a$, $u = 0$; $x \to b$, $u \to \infty$

$$x - a = \frac{(b-a)u}{1+u},\quad b - x = \frac{b-a}{1+u},\ \mathrm{d}x = \frac{(b-a)\,\mathrm{d}u}{(1+u)^2}$$

$$\mathrm{I} = \int_0^\infty \left[\frac{(b-a)u}{1+u}\cdot\left(\frac{b-a}{1+u}\right)^2\right]^{-\frac{1}{3}}\cdot\frac{(b-a)\,\mathrm{d}u}{(1+u)^2}$$

$$= \int_0^\infty \frac{u^{-\frac{1}{3}}}{1+u}\,\mathrm{d}u$$

$$= -\int_\infty^0 \frac{3v^3}{1+v^{-3}}\,\mathrm{d}v;\ v = u^{-\frac{1}{3}}$$

$$= \int_0^\infty \frac{3}{1+v^3}\,\mathrm{d}v$$

= 3J, same as the answer in part (c)