Second Problem on Integration

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(a) Two sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers are related as follows:

$$b_1 > a_1,\ a_{n+1} = \sqrt{a_n b_n},\ b_{n+1} = \frac{a_n + b_n}{2} \quad (n \ge 1)$$

Prove that both sequences converge to the same limit ℓ, say.

(b) If $b > a > 0$, show that the function f given by $u = \text{f}(t) = \frac{ab + t^2}{2t}$, $t \in (0, \infty)$ is strictly decreasing on the interval $(0, \sqrt{ab}\,]$ and strictly increasing on the interval $[\sqrt{ab}, \infty)$. Hence find an explicit expression for each of the inverse function of f.

(c) Let $I(a, b) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}}$, $0 < a < b$.

By making the substitution $t = \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}$ and $u = \frac{ab + t^2}{2t}$, show that

$$I(a, b) = I\left(\sqrt{ab}, \frac{a+b}{2}\right).$$

(d) Let $I(a_n, b_n) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_n^2 \cos^2\theta + b_n^2 \sin^2\theta}}$ where the sequences $\{a_n\}$ and $\{b_n\}$ are given by

$$a_1 = \sqrt{ab},\ b_1 = \frac{a_1 + b_1}{2},\ a_{n+1} = \sqrt{a_n b_n},\ b_{n+1} = \frac{a_n + b_n}{2} \text{ for all } n \ge 1.$$

Using (a), show that $I(a, b) = I(a_n, b_n)$.

(e) Using (a) and (d), show that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}} = \frac{\pi}{2\ell}$.

Solution

(a) $b_{n+1} - a_{n+1} = \frac{a_n + b_n}{2} - \sqrt{a_n b_n}$

$\ge \sqrt{a_n b_n} - \sqrt{a_n b_n} = 0$

$\Rightarrow b_{n+1} \ge a_{n+1}$

$\Rightarrow b_n \ge a_n\ \forall n \ge 1$

$b_n - b_{n+1} = b_n - \frac{a_n + b_n}{2}$

$= \frac{b_n - a_n}{2} \ge 0$

$\Rightarrow b_n \ge b_{n+1}$

$a_{n+1} - a_n = \sqrt{a_n b_n} - a_n$

$= \sqrt{a_n}\left(\sqrt{b_n} - \sqrt{a_n}\right)$

$= \sqrt{a_n} \cdot \frac{b_n - a_n}{\sqrt{b_n} + \sqrt{a_n}} \ge 0$

$\Rightarrow a_{n+1} \ge a_n$

$\therefore b_1 > \cdots > b_n > b_{n+1} > a_{n+1} > a_n > \cdots > a_1$.

The sequences $\{a_n\}$ is monotonic increasing and bounded above by b_1 and $\{b_n\}$ is monotonic decreasing and bounded below by a_1.

$\therefore$ Both sequences converge.

Let $\lim_{n\to\infty} a_n = k$, $\lim_{n\to\infty} b_n = m$

$b_{n+1} = \frac{a_n + b_n}{2} \Rightarrow \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} \frac{a_n + b_n}{2}$

$m = \frac{k+m}{2} \Rightarrow k = m$, let the common limit be ℓ.

(b) $f(t) = \frac{ab + t^2}{2t} = \frac{ab}{2t} + \frac{t}{2}$

$f'(t) = -\frac{ab}{2t^2} + \frac{1}{2}$; let $f'(t) = 0$, $t^2 = ab$, $t = \sqrt{ab}$

$f'(t) = \frac{1}{2t^2}\left(t + \sqrt{ab}\right)\left(t - \sqrt{ab}\right)$

If $t \in (0, \sqrt{ab}\,]$, $f'(t) < 0 \Rightarrow f(t)$ is strictly decreasing.

If $t \in [\sqrt{ab}, \infty)$, $f'(t) > 0 \Rightarrow f(t)$ is strictly increasing.

$u = \frac{ab + t^2}{2t}$

$t^2 - 2ut + ab = 0$

$t = u \pm \sqrt{u^2 - ab}$

(c) $I(a, b) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}$, $0 < a < b$.

$t = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} \Rightarrow t^2 = a^2\cos^2\theta + b^2\sin^2\theta$

$2t\,dt = (-2a^2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta)\,d\theta \Rightarrow d\theta = \frac{t\,dt}{\left(b^2 - a^2\right)\sin\theta\cos\theta}$

$\theta = 0$, $t = a$; $\theta = \frac{\pi}{2}$, $t = b$.

$t^2 - a^2 = b^2\sin^2\theta - a^2\sin^2\theta = (b^2 - a^2)\sin^2\theta \Rightarrow \sin\theta = \frac{\sqrt{t^2 - a^2}}{\sqrt{b^2 - a^2}}$

$b^2 - t^2 = b^2\cos^2\theta - a^2\cos^2\theta = (b^2 - a^2)\cos^2\theta \Rightarrow \cos\theta = \frac{\sqrt{b^2 - t^2}}{\sqrt{b^2 - a^2}}$

$I(a, b) = \int_a^b \frac{t\,dt}{t\left(b^2 - a^2\right)\cdot\sqrt{\frac{t^2 - a^2}{b^2 - a^2}}\cdot\sqrt{\frac{b^2 - t^2}{b^2 - a^2}}} = \int_a^b \frac{dt}{\sqrt{\left(b^2 - t^2\right)\left(t^2 - a^2\right)}}$ …… (*)

$I(a, b) = \int_a^{\sqrt{ab}} \frac{dt}{\sqrt{\left(b^2 - t^2\right)\left(t^2 - a^2\right)}} + \int_{\sqrt{ab}}^b \frac{dt}{\sqrt{\left(b^2 - t^2\right)\left(t^2 - a^2\right)}}$

$u = \frac{ab + t^2}{2t}$, $t = u - \sqrt{u^2 - ab}$ for $a \le t \le \sqrt{ab}$; $t = u + \sqrt{u^2 - ab}$ for $\sqrt{ab} \le t \le b$.

$t = a$, $u = \frac{a+b}{2}$; $t = \sqrt{ab}$, $u = \sqrt{ab}$; $t = b$, $u = \frac{a+b}{2}$.

When $t = u - \sqrt{u^2 - ab}$, $dt = du - \frac{u\,du}{\sqrt{u^2 - ab}} = -\frac{\left(u - \sqrt{u^2 - ab}\right)du}{\sqrt{u^2 - ab}} = -\frac{t\,du}{\sqrt{u^2 - ab}}$;

when $t = u + \sqrt{u^2 - ab}$, $dt = du + \frac{u\,du}{\sqrt{u^2 - ab}} = \frac{\left(u + \sqrt{u^2 - ab}\right)du}{\sqrt{u^2 - ab}} = \frac{t\,du}{\sqrt{u^2 - ab}}$.

$$(b^2 - t^2)(t^2 - a^2) = b^2 t^2 + a^2 t^2 - a^2 b^2 - t^4$$
$$= t^2\left[a^2 + b^2 - 4\left(\frac{a^2b^2 + t^4}{4t^2}\right)\right]$$
$$= t^2\left[a^2 + b^2 + 2ab - 4\left(\frac{ab + t^2}{2t}\right)^2\right]$$
$$= t^2[(a + b)^2 - 4u^2]$$

$$I(a, b) = \int_a^{\sqrt{ab}} \frac{\mathrm{d}t}{\sqrt{(b^2 - t^2)(t^2 - a^2)}} + \int_{\sqrt{ab}}^{b} \frac{\mathrm{d}t}{\sqrt{(b^2 - t^2)(t^2 - a^2)}}$$
$$= \int_{\frac{a+b}{2}}^{\sqrt{ab}} \frac{-\frac{t\mathrm{d}u}{\sqrt{u^2 - ab}}}{\sqrt{t^2\left[(a+b)^2 - 4u^2\right]}} + \int_{\sqrt{ab}}^{\frac{a+b}{2}} \frac{\frac{t\mathrm{d}u}{\sqrt{u^2 - ab}}}{\sqrt{t^2\left[(a+b)^2 - 4u^2\right]}}$$
$$= \int_{\sqrt{ab}}^{\frac{a+b}{2}} \frac{\mathrm{d}u}{\sqrt{u^2 - ab}\sqrt{\left[(a+b)^2 - 4u^2\right]}} + \int_{\sqrt{ab}}^{\frac{a+b}{2}} \frac{\mathrm{d}u}{\sqrt{u^2 - ab}\sqrt{\left[(a+b)^2 - 4u^2\right]}}$$
$$= 2\int_{\sqrt{ab}}^{\frac{a+b}{2}} \frac{\mathrm{d}u}{\sqrt{u^2 - ab}\sqrt{4\left[\left(\frac{a+b}{2}\right)^2 - u^2\right]}}$$
$$= \int_{\sqrt{ab}}^{\frac{a+b}{2}} \frac{\mathrm{d}u}{\sqrt{\left(\frac{a+b}{2}\right)^2 - u^2} \cdot \sqrt{u^2 - \left(\sqrt{ab}\right)^2}}$$
$$= I\left(\sqrt{ab}, \frac{a+b}{2}\right) \text{ by the formula (*).}$$

(d) $I(a_n, b_n) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{a_n^2 \cos^2\theta + b_n^2 \sin^2\theta}}$; $a_1 = \sqrt{ab}$, $b_1 = \frac{a_1 + b_1}{2}$, $a_{n+1} = \sqrt{a_n b_n}$, $b_{n+1} = \frac{a_n + b_n}{2}$, $n \ge 1$.

To show $I(a, b) = I(a_n, b_n)$ by mathematical induction on n.

By (c), $I(a, b) = I\left(\sqrt{ab}, \frac{a+b}{2}\right)$

$= I(a_1, b_1)$

The formula is true for $n = 1$.

Suppose $I(a, b) = I(a_k, b_k)$ for some positive integer k.

Use the result of (c) and replace a by a_k, b by b_k.

$I(a_k, b_k) = I\left(\sqrt{a_k b_k}, \frac{a_k + b_k}{2}\right)$

$= I(a_{k+1}, b_{k+1})$ by the definition.

$\therefore I(a, b) = I(a_{k+1}, b_{k+1})$ by induction assumption.

The formula is also true for $n = k + 1$ if it is true for $n = k$.

By the principle of mathematical induction, $I(a, b) = I(a_n, b_n)$ for all positive integer n.

(e) $$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}} = I(a, b) = I(a_n, b_n) = I\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right) = I(\ell, \ell)$$
$$= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{\ell^2 \cos^2\theta + \ell^2 \sin^2\theta}} = \frac{1}{\ell}\int_0^{\frac{\pi}{2}} \mathrm{d}\theta = \frac{\pi}{2\ell}.$$