Examples on reduction formulae

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(a) For any non-negative integers m and n, let $B(m, n) = \int_0^1 x^m (1-x)^n \, dx$.

Show that $B(m, n) = \dfrac{n}{m+1} B(m+1, n-1)$ for any $m \geq 0$, $n \geq 1$.

Hence, or otherwise, deduce that $B(m, n) = \dfrac{m!n!}{(m+n+1)!}$.

(b) (i) Evaluate $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$.

(ii) Using (b)(i) and (a), show that $\dfrac{1}{1260} \leq \dfrac{22}{7} - \pi \leq \dfrac{1}{630}$.

(a) $B(m,n) = \displaystyle\int_0^1 x^m (1-x)^n \, dx = \frac{1}{m+1}\int_0^1 (1-x)^n \, dx^{m+1}$

$$= \frac{1}{m+1}\left[(1-x)^n x^{m+1}\Big|_0^1 - \int_0^1 x^{m+1} d(1-x)^n \right]$$

$$= \frac{n}{m+1}\int_0^1 x^{m+1}(1-x)^{n-1} dx = \frac{n}{m+1} B(m+1, n-1).$$

$$B(m, n) = \frac{n}{m+1} B(m+1, n-1) = \frac{n}{m+1}\cdot\frac{n-1}{m+2} B(m+2, n-2) = \cdots$$

$$= \frac{n}{m+1}\cdot\frac{n-1}{m+2}\cdots\frac{1}{m+n} B(m+n, 0) = \frac{m!n!}{(m+n)!}\int_0^1 x^{m+n} dx \ \frac{m!n!}{(m+n)!}\cdot\frac{x}{(m+n+1)}\Bigg|_0^1$$

$$B(m, n) = \frac{m!n!}{(m+n+1)!}.$$

(b) (i) $x^4(1-x)^4 = x^4(1 - 4x + 6x^2 - 4x^3 + x^4) = x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$.

$$
\begin{array}{r l}
 & x^6 - 4x^5 + 5x^4 - 4x^2 + 4 \\
\hline
x^2 + 1 \,\big) & x^8 - 4x^7 + 6x^6 - 4x^5 + x^4 \\
 & x^8 \qquad\quad + x^6 \\
\hline
 & \quad - 4x^7 \qquad - 4x^5 \\
 & \quad - 4x^7 \qquad - 4x^5 \\
\hline
 & \qquad\qquad 5x^6 \qquad + x^4 \\
 & \qquad\qquad 5x^6 \qquad + 5x^4 \\
\hline
 & \qquad\qquad\qquad\quad - 4x^4 \\
 & \qquad\qquad\qquad\quad - 4x^4 - 4x^2 \\
\hline
 & \qquad\qquad\qquad\qquad\quad 4x^2 \\
 & \qquad\qquad\qquad\qquad\quad 4x^2 + 4 \\
\hline
 & \qquad\qquad\qquad\qquad\qquad - 4
\end{array}
$$

$$\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \right) dx$$

$$= \left(\frac{1}{7}x^7 - \frac{4}{6}x^6 + x^5 - \frac{4}{3}x^3 + 4x - 4\tan^{-1} x \right)\Bigg|_0^1$$

$$= \frac{22}{7} - \pi$$

(ii) $\frac{x^4(1-x)^4}{2} \le \frac{x^4(1-x)^4}{1+x^2} \le x^4(1-x)^4$ for $0 \le x \le 1$

$$\int_0^1 \frac{x^4(1-x)^4}{2}\,\mathrm{d}x \le \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x \le \int_0^1 x^4(1-x)^4\,\mathrm{d}x$$

$$\frac{1}{2}B(4,4) \le \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x \le B(4,4)$$

$$\frac{1}{1260} = \frac{1}{2}\cdot\frac{4!4!}{9!} = \frac{1}{2}B(4,4) \le \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x \le \frac{4!4!}{9!} = \frac{1}{630}$$

$$\frac{1}{1260} \le \frac{22}{7} - \pi \le \frac{1}{630}.$$

2. For any positive integers m, $n > 1$, let $J_{(m,n)} = \frac{m+n+1}{2}\int_0^{\frac{1}{2}} x^m (1-x)^n \,dx$.

(a) Find $J_{(n-1,n)}$ in terms of $J_{(n,n-1)}$.

(b) Using (a) or otherwise, find $J_{(n-1,n)} + J_{(n,n-1)}$ in terms of n.

(c) Hence deduce the value of $\lim_{n\to\infty} J_{(n-1,n)}$.

(a) $$J_{(n-1,n)} = \frac{n-1+n+1}{2}\int_0^{\frac{1}{2}} x^{n-1}(1-x)^n \,dx$$
$$= n\int_0^{\frac{1}{2}} x^{n-1}(1-x)^n \,dx$$
$$= n\int_0^{\frac{1}{2}} (1-x)^n \,d\left(\frac{x^n}{n}\right)$$
$$= x^n(1-x)^n\Big|_0^{\frac{1}{2}} + n\int_0^{\frac{1}{2}} x^n (1-x)^{n-1}\,dx$$
$$= \frac{1}{2^{2n}} + J_{(n,n-1)}$$

(b) $$J_{(n-1,n)} + J_{(n,n-1)} = n\int_0^{\frac{1}{2}} x^{n-1}(1-x)^n \,dx + n\int_0^{\frac{1}{2}} x^n(1-x)^{n-1}\,dx$$
$$= n\int_0^{\frac{1}{2}} \left[x^{n-1}(1-x)^n + x^n(1-x)^{n-1}\right] dx$$
$$= n\int_0^{\frac{1}{2}} x^{n-1}(1-x)^{n-1}(1-x+x)\,dx$$
$$= n\int_0^{\frac{1}{2}} x^{n-1}(1-x)^{n-1}\,dx$$

Let $I_{n-1} = \int_0^{\frac{1}{2}} x^{n-1}(1-x)^{n-1}\,dx$ and $u = 1 - x$, then $x = 1 - u$

When $x = 0$, $u = 1$; when $x = \frac{1}{2}$, $u = \frac{1}{2}$; $dx = -du$

$$I_{n-1} = \int_0^{\frac{1}{2}} x^{n-1}(1-x)^{n-1}\,dx = \int_1^{\frac{1}{2}} (1-u)^{n-1}u^{n-1}(-du)$$
$$= \int_{\frac{1}{2}}^1 (1-u)^{n-1}u^{n-1}du = \int_{\frac{1}{2}}^1 x^{n-1}(1-x)^{n-1}\,dx$$
$$2I_{n-1} = \int_0^{\frac{1}{2}} x^{n-1}(1-x)^{n-1}\,dx + \int_{\frac{1}{2}}^1 x^{n-1}(1-x)^{n-1}\,dx$$
$$= \int_0^1 x^{n-1}(1-x)^{n-1}\,dx = B(n-1, n-1)$$
$$= \frac{(n-1)!(n-1)!}{(n-1+n-1+1)!}$$ by the result of example 1(a)

$$I_{n-1} = \frac{\left[(n-1)!\right]^2}{2(2n-1)!}$$

$$J_{(n-1,n)} + J_{(n,n-1)} = n\frac{\left[(n-1)!\right]^2}{2(2n-1)!} = \frac{(n-1)!n!}{2(2n-1)!}$$

(c) $J_{(n-1,n)} + J_{(n,n-1)} = J_{(n-1,n)} + J_{(n-1,n)} - \frac{1}{2^{2n}} = \frac{(n-1)!n!}{2(2n-1)!}$

$2\,J_{(n-1,n)} = \frac{1}{2^{2n}} + \frac{(n-1)!n!}{2(2n-1)!}$

$J_{(n-1,n)} = \frac{1}{2^{2n+1}} + \frac{(n-1)!n!}{4(2n-1)!}$

$\lim_{n\to\infty} J_{(n-1,n)} = \lim_{n\to\infty}\left[\frac{1}{2^{2n+1}} + \frac{1\times 2\times\cdots\times(n-1)\times 1\times 2\times\cdots\times n}{4\times 1\times 2\times\cdots\times(n-1)\times n\times(n+1)\times\cdots\times(2n-1)}\right]$

$= \frac{1}{4}\lim_{n\to\infty}\frac{1\times 2\times\cdots\times(n-1)}{(n+1)\times\cdots\times(2n-1)}$

Let $a_n = \frac{1\times 2\times\cdots\times(n-1)}{(n+1)\times\cdots\times(2n-1)}$, then $a_n > 0$

$\frac{a_n}{a_{n+1}} = \frac{1\times 2\times\cdots\times(n-1)}{(n+1)\times\cdots\times(2n-1)} \div \frac{1\times 2\times\cdots\times(n-1)\times n}{(n+2)\times\cdots\times(2n-1)\times 2n\times(2n+1)}$

$= \frac{2n(2n+1)}{n(n+1)} = 2\times\frac{2n+1}{n+1} > 1$

$\therefore\ a_n > a_{n+1}$

$\{a_n\}$ is monotonic decreasing which is bounded below

By monotonic convergent theorem, $\lim_{n\to\infty} a_n$ exists.

Let $\lim_{n\to\infty} a_n = m$

If $m \neq 0$, then $1 = \frac{m}{m} = \lim_{n\to\infty}\frac{a_n}{a_{n+1}} = \lim_{n\to\infty} 2\times\frac{2n+1}{n+1} = 2\lim_{n\to\infty}\frac{2+\frac{1}{n}}{1+\frac{1}{n}} = 4$, which is a contradiction

$\therefore\ \lim_{n\to\infty} a_n = 0$

Consequently, $\lim_{n\to\infty} J_{(n-1,n)} = 0$.