Integration Example - symmetry

Cambridge University Mathematics Entrance Examination 2015 Q6

Created by Mr. Francis Hung on 20230226

(i) Show that $\sec^2\left(\frac{1}{4}\pi - \frac{1}{2}x\right) = \frac{2}{1+\sin x}$.

Hence integrate $\frac{1}{1+\sin x}$ with respect to x.

(ii) By means of substitution $y = \pi - x$, show that $\int_0^\pi x\mathrm{f}(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi \mathrm{f}(\sin x)\,\mathrm{d}x$,

where f is any function for which these integrals exist.

Hence evaluate $\int_0^\pi \frac{x}{1+\sin x}\,\mathrm{d}x$.

(iii) Evaluate $\int_0^\pi \frac{2x^3 - 3\pi x^2}{(1+\sin x)^2}\,\mathrm{d}x$.

(i) $\sec^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)=\frac{1}{\cos^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)}$

$=\frac{1}{\frac{1}{2}\left[1+\cos 2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\right]}$ (double angle formula $\cos 2\theta = 2\cos^2\theta - 1$)

$=\frac{2}{1+\cos\left(\frac{1}{2}\pi-x\right)}=\frac{2}{1+\sin x}$

$$\int\frac{1}{1+\sin x}\mathrm{d}x=\frac{1}{2}\int\sec^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\mathrm{d}x$$

$$=-\int\sec^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\mathrm{d}\left(\frac{1}{4}\pi-\frac{1}{2}x\right)$$

$$=-\tan\left(\frac{1}{4}\pi-\frac{1}{2}x\right)+C$$, where C is a constant

(ii) Let $y = \pi - x$, then $x = \pi - y$; $\mathrm{d}x = -\mathrm{d}y$; when $x = 0$, $y = \pi$; when $x = \pi$, $y = 0$.

$$\int_0^\pi x\mathrm{f}(\sin x)\mathrm{d}x=\int_\pi^0(\pi-y)\mathrm{f}(\sin(\pi-y))(-\mathrm{d}y)$$

$$=\int_0^\pi(\pi-y)\mathrm{f}(\sin y)\mathrm{d}y$$

$$=\int_0^\pi\pi\mathrm{f}(\sin y)\mathrm{d}y-\int_0^\pi y\mathrm{f}(\sin y)\mathrm{d}y$$

$$=\int_0^\pi\pi\mathrm{f}(\sin x)\mathrm{d}x-\int_0^\pi x\mathrm{f}(\sin x)\mathrm{d}x$$

$$2\int_0^\pi x\mathrm{f}(\sin x)\mathrm{d}x=\pi\int_0^\pi\mathrm{f}(\sin x)\mathrm{d}x$$

$$\int_0^\pi x\mathrm{f}(\sin x)\mathrm{d}x=\frac{\pi}{2}\int_0^\pi\mathrm{f}(\sin x)\mathrm{d}x$$

Let $\mathrm{f}(x)=\frac{1}{1+\sin x}$, then by the above result,

$$\int_0^\pi\frac{x}{1+\sin x}\mathrm{d}x=\frac{\pi}{2}\int_0^\pi\frac{1}{1+\sin x}\mathrm{d}x$$

$$=\frac{\pi}{2}\left(-\tan\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\right)\Bigg|_0^\pi$$ by the result of (a)

$$=-\frac{\pi}{2}\left(\tan\left(-\frac{\pi}{4}\right)-\tan\frac{\pi}{4}\right)=\pi$$

(iii) Consider $I=\int_0^{\pi}\frac{x^3}{(1+\sin x)^2}\,\mathrm{d}x$.

Let $y = \pi - x$, then $x = \pi - y$; $\mathrm{d}x = -\mathrm{d}y$; when $x = 0$, $y = \pi$; when $x = \pi$, $y = 0$.

$$I=\int_\pi^0\frac{(\pi-y)^3}{\left[1+\sin(\pi-y)\right]^2}(-\mathrm{d}y)=\int_0^\pi\frac{\pi^3-3\pi^2y+3\pi y^2-y^3}{(1+\sin y)^2}\mathrm{d}y=\int_0^\pi\frac{\pi^3-3\pi^2x+3\pi x^2-x^3}{(1+\sin x)^2}\mathrm{d}x$$

$$\int_0^\pi\frac{x^3}{(1+\sin x)^2}\mathrm{d}x+\int_0^\pi\frac{x^3}{(1+\sin x)^2}\mathrm{d}x-\int_0^\pi\frac{3\pi x^2}{(1+\sin x)^2}\mathrm{d}x=\int_0^\pi\frac{\pi^3-3\pi^2x}{(1+\sin x)^2}\mathrm{d}x$$

$$\int_0^\pi\frac{2x^3-3\pi x^2}{(1+\sin x)^2}\mathrm{d}x=\pi^3\int_0^\pi\frac{1}{(1+\sin x)^2}\mathrm{d}x-3\pi^2\int_0^\pi\frac{x}{(1+\sin x)^2}\mathrm{d}x$$

$$=\pi^3\int_0^\pi\frac{1}{(1+\sin x)^2}\mathrm{d}x-3\pi^2\cdot\frac{\pi}{2}\int_0^\pi\frac{1}{(1+\sin x)^2}\mathrm{d}x \quad \text{by the result of (ii)}$$

$$=-\frac{\pi^3}{2}\int_0^\pi\frac{1}{(1+\sin x)^2}\mathrm{d}x$$

$$=-\frac{\pi^3}{8}\int_0^\pi\sec^4\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\mathrm{d}x \quad \text{by the result of (i)}$$

$$=-\frac{\pi^3}{8}\int_0^\pi\sec^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\left(1+\tan^2\left(\frac{1}{4}\pi-\frac{1}{2}x\right)\right)\mathrm{d}x$$

$$=\frac{\pi^3}{4}\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\sec^2u\left(1+\tan^2u\right)\mathrm{d}u,\ u=\frac{\pi}{4}-\frac{x}{2};\ x=0,\ u=\frac{\pi}{4};\ x=\pi,\ u=-\frac{\pi}{4};\ \mathrm{d}x=-2\mathrm{d}u$$

$$=-\frac{\pi^3}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(1+\tan^2u\right)\mathrm{d}(\tan u)=-\frac{\pi^3}{4}\left(\tan u+\frac{1}{3}\tan^3u\right)_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$=-\frac{\pi^3}{4}\left[\left(1+\frac{1}{3}\right)-\left(-1-\frac{1}{3}\right)\right]$$

$$=-\frac{2\pi^3}{3}$$