Gramma function

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Calculus by Michael Spivak p.328-329 Q26-Q27

The following two questions guide you to find $\int_0^\infty e^{-y^2}\,\mathrm{d}y = \frac{\sqrt{\pi}}{2}$.

Q 26 (a) Use the reduction formula for $\int \sin^n x\mathrm{d}x$ to show that

$$\int_0^{\pi/2} \sin^n x\mathrm{d}x = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\mathrm{d}x.$$

(b) Now show that

$$\int_0^{\pi/2} \sin^{2n+1} x\mathrm{d}x = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n+1},$$

$$\int_0^{\pi/2} \sin^{2n} x\mathrm{d}x = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\cdots\cdot\frac{2n-1}{2n},$$

and conclude that

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\frac{\int_0^{\pi/2}\sin^{2n} x\mathrm{d}x}{\int_0^{\pi/2}\sin^{2n+1} x\mathrm{d}x}.$$

(c) Using the fact that $0 < \sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x$ for $0 < x < \frac{\pi}{2}$,

show that $1 < \frac{\int_0^{\pi/2}\sin^{2n} x\mathrm{d}x}{\int_0^{\pi/2}\sin^{2n+1} x\mathrm{d}x} < 1 + \frac{1}{2n}$;

hence show that $\lim_{n\to\infty}\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right) = \frac{\pi}{2}$

(d) Show also that $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n}}\cdot\frac{2\cdot4\cdot6\cdot\cdots\cdot2n}{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}\right] = \sqrt{\pi}$

Q27 (a) Show that $\int_0^1 \left(1-x^2\right)^n \mathrm{d}x = \frac{2}{3}\cdot\frac{4}{5}\cdot\cdots\cdot\frac{2n}{2n+1}$,

and $\int_0^\infty \frac{1}{\left(1+x^2\right)^n}\mathrm{d}x = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{2n-3}{2n-2}$.

(b) Prove, using the derivative, that

$1 - x^2 \le e^{-x^2}$ for $0 \le x \le 1$,

and $e^{-x^2} \le \frac{1}{1+x^2}$ for $0 \le x$.

(c) Integrate the n^{th} powers of these inequalities from 0 to 1 and from 0 to ∞, respectively. Then use the substitution $y = \sqrt{n}x$ to show that

$$\sqrt{n}\frac{2}{3}\cdot\frac{4}{5}\cdot\cdots\cdot\frac{2n}{2n+1} \le \int_0^{\sqrt{n}} e^{-y^2}\mathrm{d}y \le \int_0^\infty e^{-y^2}\mathrm{d}y \le \frac{\pi}{2}\cdot\sqrt{n}\frac{1}{2}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{2n-3}{2n-2}.$$

(d) Use Problem 26(c) to show that $\int_0^\infty e^{-y^2}\mathrm{d}y = \frac{\sqrt{\pi}}{2}$.

Q26 (a) $\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \sin^{n-1} x \cdot \sin x dx$

$$= -\int_0^{\pi/2} \sin^{n-1} x d(\cos x)$$

$$= -\sin^{n-1} x \cos x \Big|_0^{\pi/2} + \int_0^{\pi/2} \cos x d(\sin^{n-1} x)$$

$$= (n-1)\int_0^{\pi/2} \cos^2 x \cdot \sin^{n-2} x dx$$

$$= (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2} x dx$$

$$= (n-1)\left[\int_0^{\pi/2} \sin^{n-2} x dx - \int_0^{\pi/2} \sin^n x dx\right]$$

$$\int_0^{\pi/2} \sin^n x dx = (n-1)\int_0^{\pi/2} \sin^{n-2} x dx - (n-1)\int_0^{\pi/2} \sin^n x dx$$

$$n\int_0^{\pi/2} \sin^n x dx = (n-1)\int_0^{\pi/2} \sin^{n-2} x dx$$

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

(b) $\int_0^{\pi/2} \sin^{2n+1} x dx = \frac{2n}{2n+1}\int_0^{\pi/2} \sin^{2n-1} x dx$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\int_0^{\pi/2} \sin^{2n-3} x dx$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\cdots\frac{2}{3}\cdot\int_0^{\pi/2} \sin x dx$$

$$= \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\cdots\frac{2n}{2n+1}$$

$$\int_0^{\pi/2} \sin^{2n} x dx = \frac{2n-1}{2n}\int_0^{\pi/2} \sin^{2n-2} x dx$$

$$= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\int_0^{\pi/2} \sin^{2n-2} x dx$$

$$= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\cdots\frac{1}{2}\cdot\int_0^{\pi/2} \sin^0 x dx$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdots\frac{2n-1}{2n}$$

$$\frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} = \frac{\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdots\frac{2n-1}{2n}}{\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\cdots\frac{2n}{2n+1}}$$

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx}$$

(c) $0 < \sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x$ for $0 < x < \frac{\pi}{2}$.

$$0 < \int_0^{\pi/2} \sin^{2n+1} x dx < \int_0^{\pi/2} \sin^{2n} x dx < \int_0^{\pi/2} \sin^{2n-1} x dx$$

$$1 < \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} < \frac{\int_0^{\pi/2} \sin^{2n-1} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx}$$

$$1 < \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx} < \frac{\int_0^{\pi/2} \sin^{2n-1} x\,dx}{\frac{2n}{2n+1}\int_0^{\pi/2} \sin^{2n-1} x\,dx}, \text{ by (a)}$$

$$1 < \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx} < 1 + \frac{1}{2n}$$

$$\lim_{n\to\infty} 1 \le \lim_{n\to\infty} \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx} \le 1 + \lim_{n\to\infty} \frac{1}{2n} = 1$$

By squeezing principle, $\lim_{n\to\infty} \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx} = 1$.

From (b), $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx}$

$$\frac{\pi}{2} = \lim_{n\to\infty}\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right)\lim_{n\leftarrow\infty}\frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx}$$

$$\therefore \lim_{n\to\infty}\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right) = \frac{\pi}{2}$$

(d) By (b), $\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx}$

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty}\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\cdot\frac{2n}{2n-1}\cdot\frac{1}{\sqrt{2n+1}}\sqrt{\lim_{n\to\infty}\frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx}}$$

$$\sqrt{\pi} = \lim_{n\to\infty}\frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdots\cdot\frac{2n}{2n-1}\cdot\frac{\sqrt{2}}{\sqrt{2n+1}} \quad \left(\because \lim_{n\to\infty}\frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx} = 1\right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{n}}\cdot\frac{2\cdot 4\cdot 6\cdots\cdot 2n}{1\cdot 3\cdot 5\cdots\cdot(2n-1)}\cdot\sqrt{\frac{2}{2+\frac{1}{n}}}\right]$$

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n}}\cdot\frac{2\cdot 4\cdot 6\cdots\cdot 2n}{1\cdot 3\cdot 5\cdots\cdot(2n-1)}\right] = \sqrt{\pi}$$

Q27 (a) $\int_0^1 (1-x^2)^n \, dx$, let $x = \cos\theta$, $dx = -\sin\theta \, d\theta$; $x = 0$, $\theta = \frac{\pi}{2}$; $x = 1$, $\theta = 0$.

$$\int_0^1 (1-x^2)^n \, dx = -\int_{\pi/2}^0 (1-\cos^2\theta)^n \sin\theta d\theta$$
$$= \int_0^{\pi/2} \sin^{2n+1}\theta d\theta$$
$$= \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\cdots\frac{2n}{2n+1} \quad \text{by Q26(b)}$$

$\int_0^\infty \frac{1}{(1+x^2)^n} dx$, let $x = \cot\theta$, $dx = -\csc^2\theta \, d\theta$; $x \to 0^+$, $\theta \to \frac{\pi}{2}^-$; $x \to \infty$, $\theta \to 0^+$.

$$\int_0^\infty \frac{1}{(1+x^2)^n} dx = -\int_{\pi/2}^0 \frac{1}{(1+\cot^2\theta)^n} \csc^2\theta d\theta$$
$$= \int_0^{\pi/2} \frac{1}{\csc^{2n}\theta} \csc^2\theta d\theta$$
$$= \int_0^{\pi/2} \sin^{2n-2}\theta d\theta$$
$$= \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdots\frac{2n-3}{2n-2} \quad \text{by Q26(b)}$$

(b) Let $f(x) = e^{-x^2} + x^2 - 1$

$f'(x) = -2x e^{-x^2} + 2x$

$= 2x(1 - e^{-x^2})$

$f'(x) = 0 \Rightarrow x = 0$

For $0 < x$, $e^{x^2} > e^0 = 1$

$\Rightarrow e^{-x^2} < 1$

$\Rightarrow 1 - e^{-x^2} > 0$

$\Rightarrow 2x(1 - e^{-x^2}) > 0$

$\Rightarrow f'(x) > 0$

$\therefore$ f '(x) is strictly increasing for $x > 0$

$f(x) > f(0)$

$e^{-x^2} + x^2 - 1 > 0$

$1 - x^2 \le e^{-x^2}$ for $0 \le x \le 1$.

Let $g(x) = 1 + x^2 - e^{x^2}$

$g'(x) = 2x - 2x e^{x^2}$

$= 2x(1 - e^{x^2})$

$g'(0) = 0$

For $x > 0$, $e^{x^2} > e^0 = 1$

$1 - e^{x^2} < 0 \Rightarrow g'(x) < 0$

$\therefore$ g(x) is strictly decreasing for $x \ge 0$

$g(x) < g(0)$ for $x > 0$

$1 + x^2 < e^{x^2}$

$$e^{-x^2} < \frac{1}{1+x^2}$$

$$e^{-x^2} \le \frac{1}{1+x^2} \quad \text{for } 0 \le x.$$

(c) By (b), $1 - x^2 \le e^{-x^2}$ for $0 \le x \le 1$ and $e^{-x^2} \le \dfrac{1}{1+x^2}$ for $x \ge 0$.

$\Rightarrow (1 - x^2)^n \le e^{-nx^2}$ for $0 \le x \le 1$ and $e^{-nx^2} \le \dfrac{1}{\left(1+x^2\right)^n}$ for $x \ge 0$.

$\Rightarrow \int_0^1 \left(1-x^2\right)^n \mathrm{d}x \le \int_0^1 e^{-nx^2} \mathrm{d}x$ and $\int_0^\infty e^{-nx^2} \mathrm{d}x \le \int_0^\infty \dfrac{1}{\left(1+x^2\right)^n} \mathrm{d}x$ …… (*)

$y = \sqrt{n}x$, $\mathrm{d}x = \dfrac{\mathrm{d}y}{\sqrt{n}}$; $x = 0$, $y = 0$; $x = 1$, $y = \sqrt{n}$; $x \to \infty$, $y \to \infty$.

$\int_0^1 e^{-nx^2} \mathrm{d}x = \int_0^{\sqrt{n}} e^{-y^2} \dfrac{\mathrm{d}y}{\sqrt{n}}$ and $\int_0^\infty e^{-nx^2} \mathrm{d}x = \int_0^\infty e^{-y^2} \dfrac{\mathrm{d}y}{\sqrt{n}}$

$= \dfrac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} \mathrm{d}y$ and $= \dfrac{1}{\sqrt{n}} \int_0^\infty e^{-y^2} \mathrm{d}y$

By (*), $\int_0^1 \left(1-x^2\right)^n \mathrm{d}x \le \int_0^1 e^{-nx^2} \mathrm{d}x \le \int_0^\infty e^{-nx^2} \mathrm{d}x \le \int_0^\infty \dfrac{1}{\left(1+x^2\right)^n} \mathrm{d}x$

$$\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\cdots\frac{2n}{2n+1} \le \frac{1}{\sqrt{n}}\int_0^{\sqrt{n}} e^{-y^2}\mathrm{d}y \le \frac{1}{\sqrt{n}}\int_0^\infty e^{-y^2}\mathrm{d}y \le \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\cdots\frac{2n-3}{2n-2}$$

$$\sqrt{n}\,\frac{2}{3}\cdot\frac{4}{5}\cdots\cdots\frac{2n}{2n+1} \le \int_0^{\sqrt{n}} e^{-y^2}\mathrm{d}y \le \int_0^\infty e^{-y^2}\mathrm{d}y \le \frac{\pi}{2}\cdot\sqrt{n}\,\frac{1}{2}\cdot\frac{3}{4}\cdots\cdots\frac{2n-3}{2n-2} \text{ ....(**)}$$

(d) Use Problem 26(c) to show that $\int_0^\infty e^{-y^2}\mathrm{d}y = \dfrac{\sqrt{\pi}}{2}$.

(d) Take limit as $n \to \infty$ in (**)

$$\lim_{n\to\infty}\left(\sqrt{n}\,\frac{2}{3}\cdot\frac{4}{5}\cdots\cdots\frac{2n}{2n+1}\right) \le \int_0^\infty e^{-y^2}\mathrm{d}y \le \frac{\pi}{2}\cdot\lim_{n\to\infty}\left(\sqrt{n}\,\frac{1}{2}\cdot\frac{3}{4}\cdots\cdots\frac{2n-3}{2n-2}\right)$$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}\cdot\frac{2\cdot4\cdot6\cdots\cdots2n}{1\cdot3\cdot5\cdots\cdots(2n-1)}\right)\lim_{n\to\infty}\frac{n}{2n+1} \le \int_0^\infty e^{-y^2}dy \le \frac{\pi}{2}\cdot\lim_{n\to\infty}\left(\frac{\sqrt{n}}{1}\cdot\frac{1\cdot3\cdot5\cdots\cdots(2n-1)}{2\cdot4\cdot6\cdots\cdots2n}\right)\lim_{n\to\infty}\frac{2n}{2n-3}$$

$\dfrac{\sqrt{\pi}}{2} \le \int_0^\infty e^{-y^2}\mathrm{d}y \le \dfrac{\pi}{2}\cdot\dfrac{1}{\sqrt{\pi}} = \dfrac{\sqrt{\pi}}{2}$; by the result of 26(d)

By squeezing principle, $\int_0^\infty e^{-y^2}\mathrm{d}y = \dfrac{\sqrt{\pi}}{2}$.