Double Integral

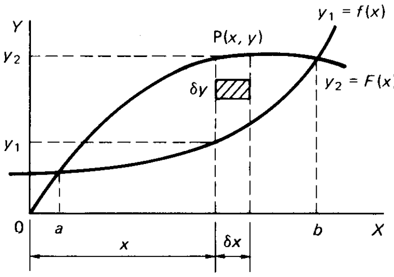
Created by Mr. Francis Hung on 20220212. Last updated: 26 February 2023

To evaluate $\mathrm{I}=\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$

$$\mathrm{I}\times\mathrm{I}=\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \times \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \quad = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x \times \int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,\mathrm{d}x\mathrm{d}y \quad = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}\,\mathrm{d}x\mathrm{d}y$$

(a) *Areas of plane figures*



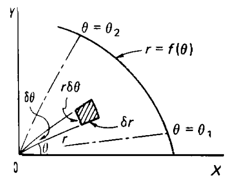
Area of element $\delta A = \delta x \delta y$

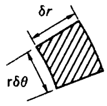
Area of strip $\doteqdot \sum_{y=y_1}^{y=y_2} \delta x\,\delta y$

Area of all such strips $\doteqdot \sum_{x=a}^{x=b}\left\{\sum_{y=y_1}^{y=y_2} \delta x\,\delta y\right\}$

If $\delta x \to 0$ and $\delta y \to 0$, $\quad \underline{A = \int_a^b \int_{y_1}^{y_2} \mathrm{d}y\,\mathrm{d}x}$

(b) *Areas of plane figures bounded by a polar curve* $r = f(\theta)$ *and* radius vectors at $\theta = \theta_1$ and $\theta = \theta_2$.





Small arc of circle of radius r, subtending angle $\delta\theta$ at centre.

$\therefore$ Arc $= r\,\delta\theta$

Area of element $\delta A \doteqdot r\,\delta\theta\,\delta r$

Area of thin sector $\doteqdot \sum_{r=0}^{r=f(\theta)} r\,\delta\theta\,\delta r$

$\therefore$ Total area of all such sectors $\doteqdot \sum_{\theta=\theta_1}^{\theta=\theta_2}\left\{\sum_{r=0}^{r=f(\theta)} r\,\delta r\,\delta\theta\right\}$

$\therefore$ If $\delta r \to 0$ and $\delta\theta \to 0$ $\quad \underline{A = \int_{\theta_1}^{\theta_2}\int_0^{r=f(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta}$

Using the transformation $x = r\cos\theta$, $y = r\sin\theta$

then $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$

when $x \to \infty$, $y \to \infty$ is equivalent to $r = 0$ to $r \to \infty$, $\theta = 0$ to $\theta = 2\pi$

$$\mathrm{I}^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\left(r^2\cos^2\theta + r^2\sin^2\theta\right)} r\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi}\mathrm{d}\theta\int_0^{\infty} e^{-r^2} r\mathrm{d}r$$

$$= 2\pi\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^{\infty} = \pi$$

$\therefore \mathrm{I} = \sqrt{\pi}$

Exercise

Show that $\int_0^{\infty}\frac{x}{1+x^3}dx = \int_0^{\infty}\frac{1}{1+x^3}dx$, and hence find the common integral.

Numerical answer: $\frac{2\sqrt{3}\pi}{9}$