ArcLength Example

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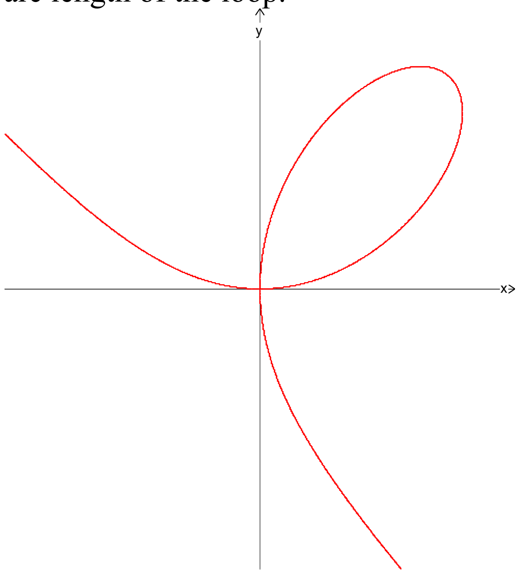
Given $x^3 + y^3 = 3axy$, find the arc length of the loop.



Let $x = r\cos\theta$, $y = r\sin\theta$

$r^3\cos^3\theta + r^3\sin^3\theta = 3ar^2\sin\theta\cos\theta$

$$r = \frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}$$

$$\frac{dr}{d\theta} = 3a\frac{(\cos^3\theta+\sin^3\theta)(\cos^2\theta-\sin^2\theta)-3\sin\theta\cos\theta(-\cos^2\theta\sin\theta+\sin^2\theta\cos\theta)}{(\cos^3\theta+\sin^3\theta)^2}$$

$$= 3a\frac{\cos^5\theta-\sin^5\theta+2\cos^3\theta\sin^2\theta-2\sin^3\theta\cos^2\theta}{(\cos^3\theta+\sin^3\theta)^2}$$

$$= 3a\frac{\cos^3\theta(\cos^2\theta+2\sin^2\theta)-\sin^3\theta(\sin^2\theta+2\cos^2\theta)}{(\cos^3\theta+\sin^3\theta)^2}$$

$$= 3a\frac{\cos^3\theta(1+\sin^2\theta)-\sin^3\theta(1+\cos^2\theta)}{(\cos^3\theta+\sin^3\theta)^2}$$

$$= 3a\frac{\cos^3\theta-\sin^3\theta+\cos^3\theta\sin^2\theta-\sin^3\theta\cos^2\theta}{(\cos\theta+\sin\theta)^2(\cos^2\theta-\sin\theta\cos\theta+\sin^2\theta)^2}$$

$$= 3a\frac{(\cos\theta-\sin\theta)(\cos^2\theta+\sin\theta\cos\theta+\sin^2\theta)+\sin^2\theta\cos^2\theta(\cos\theta-\sin\theta)}{(\cos^2\theta+2\sin\theta\cos\theta+\sin^2\theta)(1-\sin\theta\cos\theta)^2}$$

$$= 3a\frac{(\cos\theta-\sin\theta)(1+\sin\theta\cos\theta+\sin^2\theta\cos^2\theta)}{(1+2\sin\theta\cos\theta)(1-\sin\theta\cos\theta)^2}$$

Let $u = \sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \sin\theta\cos\theta = \frac{u}{2}$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left(\frac{3a\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}\right)^2 + \left[3a\frac{(\cos\theta-\sin\theta)(1+\sin\theta\cos\theta+\sin^2\theta\cos^2\theta)}{(1+2\sin\theta\cos\theta)(1-\sin\theta\cos\theta)^2}\right]^2$$

$$= \frac{\left(\frac{3au}{2}\right)^2}{(1+2\sin\theta\cos\theta)(1-\sin\theta\cos\theta)^2} + 9a^2\frac{(\cos\theta-\sin\theta)^2(1+\sin\theta\cos\theta+\sin^2\theta\cos^2\theta)^2}{(1+2\sin\theta\cos\theta)^2(1-\sin\theta\cos\theta)^4}$$

$$= 9a^2 \frac{\left(\frac{u}{2}\right)^2 (1+u)\left(1-\frac{u}{2}\right)^2 + (1-u)\left(1+\frac{u}{2}+\frac{u^2}{4}\right)^2}{(1+u)^2\left(1-\frac{u}{2}\right)^4}$$

$$= 9a^2 \frac{u^2(1+u)(2-u)^2 + (1-u)(4+2u+u^2)^2}{(1+u)^2(2-u)^4}$$

$$= 9a^2 \frac{-6u^4 - 8u^3 + 16}{(1+u)^2(2-u)^4}$$

Arc length = $\int_0^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sqrt{9a^2 \frac{-6u^4 - 8u^3 + 16}{(1+u)^2(2-u)^4}}\,\mathrm{d}\theta$

$$= 3a\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{\frac{-3\sin^4 2\theta - 4\sin^3 2\theta + 8}{(1+\sin 2\theta)^2(2-\sin 2\theta)^4}}\,\mathrm{d}\theta$$

$$= \frac{3a\sqrt{2}}{2}\int_0^{\pi} \frac{\sqrt{-3\sin^4\alpha - 4\sin^3\alpha + 8}}{(1+\sin\alpha)(2-\sin\alpha)^2}\,\mathrm{d}\alpha,\ \alpha = 2\theta$$

Let $t = \tan\frac{\alpha}{2}$, $\sin\alpha = \frac{2t}{1+t^2}$, $\mathrm{d}\alpha = \frac{2dt}{1+t^2}$; $\alpha = 0$, $t = 0$; $\alpha \to \pi$, $t \to \infty$

Arc length = $\frac{3a\sqrt{2}}{2}\int_0^{\infty} \frac{\sqrt{-3\left(\frac{2t}{1+t^2}\right)^4 - 4\left(\frac{2t}{1+t^2}\right)^3 + 8}}{\left(1+\frac{2t}{1+t^2}\right)\left(2-\frac{2t}{1+t^2}\right)^2} \cdot \frac{2\mathrm{d}t}{1+t^2}$

$$= 3a\sqrt{2}\int_0^{\infty} \frac{\sqrt{-3\cdot 16t^4 - 4\cdot 8t^3(1+t^2) + 8(1+t^2)^4}}{(1+2t+t^2)(2+2t^2-2t)^2}\,\mathrm{d}t$$

$$= 3a\int_0^{\infty} \frac{\sqrt{-6t^4 - 4t^3(1+t^2) + (1+t^2)^4}}{(1+t)^2(1-t+t^2)^2}\,\mathrm{d}t$$

$$= 3a\int_0^{\infty} \frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1+t^3)^2}\,\mathrm{d}t$$

= $3a(1.63916)$, by using the software Mathematica

= $4.91748a$