Polar area example 1

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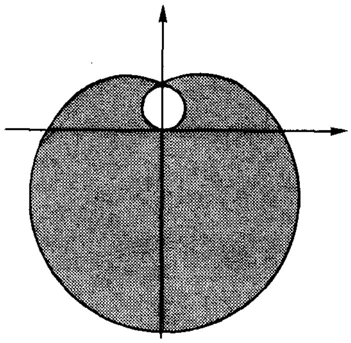
Breakthrough Calculus and Coordinate Geometry by Y.L. Ng & K. M. Pang p.243 Q20

Find the area of the shaded region bounded in the shaded area of the curve $r = a\sin^3\frac{\theta}{3}$ $(a > 0)$.



We first find the indefinite integral $\frac{1}{2}\int r^2 \mathrm{d}\theta$.

$$\frac{1}{2}\int r^2 \mathrm{d}\theta = \frac{1}{2}\int\left(a\sin^3\frac{\theta}{3}\right)^2 \mathrm{d}\theta$$

$$= \frac{3a^2}{2}\int\left(\sin^3\frac{\theta}{3}\right)^2 \mathrm{d}\frac{\theta}{3}$$

$$= \frac{3a^2}{2}\int\left(\sin^3\alpha\right)^2 \mathrm{d}\alpha,\quad \alpha = \frac{\theta}{3}$$

$$= \frac{3a^2}{2}\int\left(\frac{3\sin\alpha - \sin 3\alpha}{4}\right)^2 \mathrm{d}\alpha,\ \because \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

$$= \frac{3a^2}{32}\int\left(9\sin^2\alpha - 6\sin\alpha\sin 3\alpha + \sin^2 3\alpha\right)\mathrm{d}\alpha$$

$$= \frac{3a^2}{32}\int\left[\frac{9}{2}(1-\cos 2\alpha) + 3(\cos 4\alpha - \cos 2\alpha) + \frac{1}{2}(1-\cos 6\alpha)\right]\mathrm{d}\alpha$$

$$= \frac{3a^2}{32}\int\left(5 - \frac{9}{2}\cos 2\alpha + 3\cos 4\alpha - 3\cos 2\alpha - \frac{1}{2}\cos 6\alpha\right)\mathrm{d}\alpha$$

$$= \frac{3a^2}{32}\int\left(5 - \frac{15}{2}\cos 2\alpha + 3\cos 4\alpha - \frac{1}{2}\cos 6\alpha\right)\mathrm{d}\alpha$$

$$= \frac{3a^2}{32}\left(5\alpha - \frac{15}{4}\sin 2\alpha + \frac{3}{4}\sin 4\alpha - \frac{1}{12}\sin 6\alpha\right) + c\text{, where } c \text{ is a constant}$$

$$= \frac{3a^2}{32}\left(\frac{5\theta}{3} - \frac{15}{4}\sin\frac{2\theta}{3} + \frac{3}{4}\sin\frac{4\theta}{3} - \frac{1}{12}\sin 2\theta\right) + c$$

$$= \frac{a^2}{128}\left(20\theta - 45\sin\frac{2\theta}{3} + 9\sin\frac{4\theta}{3} - \sin 2\theta\right) + c$$

Shaded area = $2\left[\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\left(a\sin^3\frac{\theta}{3}\right)^2 d\theta \;-\; \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(a\sin^3\frac{\theta}{3}\right)^2 d\theta\right]$

$$=\frac{a^2}{64}\left(20\theta-45\sin\frac{2\theta}{3}+9\sin\frac{4\theta}{3}-\sin 2\theta\right)\Bigg|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}-\frac{a^2}{64}\left(20\theta-45\sin\frac{2\theta}{3}+9\sin\frac{4\theta}{3}-\sin 2\theta\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$=\frac{a^2}{64}(30\pi-45\sin\pi+9\sin 2\pi-\sin 6\pi)-2\left[\frac{a^2}{64}\left(10\pi-45\sin\frac{\pi}{3}+9\sin\frac{2\pi}{3}-\sin\pi\right)\right]$$

$$=\frac{a^2}{64}\left(30\pi-20\pi+45\sqrt{3}-9\sqrt{3}\right)$$

$$=\frac{a^2}{64}\left(10\pi+36\sqrt{3}\right)$$

$$=\frac{a^2}{32}\left(5\pi+18\sqrt{3}\right)$$