Polar area example 2

1993 HKAL Pure Mathematics Paper 2 Q10 (c)

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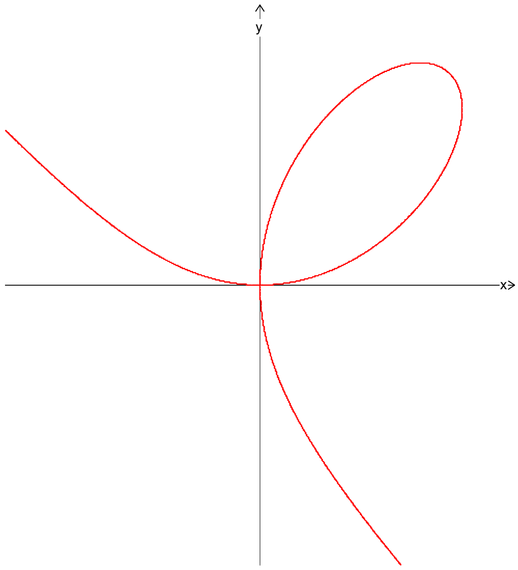
Given $x^3 + y^3 = 3axy$, find the area of the loop.



$x = r\cos\theta$, $y = r\sin\theta$, sub. into $x^3 + y^3 = 3axy$

$(r\cos\theta)^3 + (r\sin\theta)^3 = 3a(r\cos\theta)(r\sin\theta)$

$\therefore$ The polar equation of Γ is $r = \dfrac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$

$\dfrac{3a\sin(\pi+\theta)\cos(\pi+\theta)}{\sin^3(\pi+\theta)+\cos^3(\pi+\theta)} = \dfrac{-3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta} \Rightarrow$ the curve repeats for every multiple of π.

$$r = \frac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta} = \frac{3a\sin\theta\cos\theta}{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)} = \frac{\frac{3a}{2}\sin 2\theta}{\sqrt{2}\sin(\theta+\frac{\pi}{4})(1-\frac{1}{2}\sin 2\theta)}$$

$1 - \dfrac{1}{2}\sin 2\theta \ge 1 - \dfrac{1}{2} = \dfrac{1}{2} > 0$

r is undefined when $\theta = \dfrac{3\pi}{4}$.

When $\theta \to \left(\dfrac{3\pi}{4}\right)^-$, $r \to -\infty$; when $\theta \to \left(\dfrac{3\pi}{4}\right)^+$, $r \to \infty$.

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\left(\frac{3\pi}{4}\right)^-$	$\left(\frac{3\pi}{4}\right)^+$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	π
	0°	15°	30°	45°	60°	75°	90°	105°	120°	134°	136°	150°	165°	180°
r	0	$0.8a$	$1.7a$	$2.1a$	$1.7a$	$0.8a$	0	$-0.8a$	$-2.5a$	$-40a$	$40a$	$2.5a$	$0.8a$	0

From the table, the small loop corresponds to $0 \le \theta \le \dfrac{\pi}{2}$.

$$\text{Area} = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}\right)^2 d\theta = 9a^2\int_0^{\frac{\pi}{4}}\left(\frac{\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}\right)^2 d\theta$$

$$= 9a^2\int_0^{\frac{\pi}{4}} \frac{\tan\theta\sec\theta}{(\tan^3\theta+1)^2} d\theta = 3a^2\int_1^2 \frac{dw}{w^2}, \quad w = 1 + \tan^3\theta,\ dw = 3\tan^2\theta\sec^2\theta\, d\theta$$

$$= 3a^2\left(-\frac{1}{w}\right)_1^2 = 3a^2\left(-\frac{1}{2}+1\right) = \frac{3a^2}{2} \text{ sq. units}$$

Method 2

Perform the rotation of axis (in anti-clockwise direction by 45°): $\begin{cases} x = \dfrac{1}{\sqrt{2}}(x_1 - y_1) \\ y = \dfrac{1}{\sqrt{2}}(x_1 + y_1) \end{cases}$

$$\left[\frac{1}{\sqrt{2}}(x_1 - y_1)\right]^3 + \left[\frac{1}{\sqrt{2}}(x_1 + y_1)\right]^3 = 3a\frac{1}{\sqrt{2}}(x_1 - y_1)\frac{1}{\sqrt{2}}(x_1 + y_1)$$

$$\frac{1}{2\sqrt{2}}\left(2x_1^3 + 6x_1y_1^2\right) = \frac{3a}{2}\left(x_1^2 - y_1^2\right)$$

$$2x_1^3 + 6x_1y_1^2 = 3a\sqrt{2}\left(x_1^2 - y_1^2\right)$$

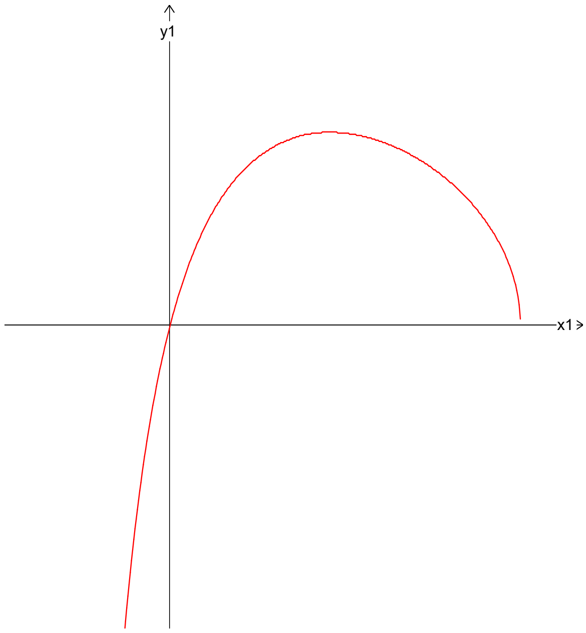
$$\left(6x_1 + 3a\sqrt{2}\right)y_1^2 = 3a\sqrt{2}x_1^2 - 2x_1^3$$

$$y_1^2 = \frac{3a\sqrt{2}x_1^2 - 2x_1^3}{6x_1 + 3a\sqrt{2}}$$

$$y_1^2 = \frac{\left(3a\sqrt{2} - 2x_1\right)x_1^2}{3\left(2x_1 + a\sqrt{2}\right)}$$

$$y_1^2 = \frac{\left(3a - \sqrt{2}x_1\right)x_1^2}{3\left(\sqrt{2}x_1 + a\right)}$$

$$y_1 = \pm\frac{x_1}{\sqrt{3}} \cdot \sqrt{\frac{3a - \sqrt{2}x_1}{\sqrt{2}x_1 + a}}$$



After rotation, the area is $2\times\int_0^{\frac{3a}{\sqrt{2}}} y_1 \mathrm{d}x_1$

$$= 2\int_0^{\frac{3a}{\sqrt{2}}} \frac{x}{\sqrt{3}}\cdot\sqrt{\frac{3a-\sqrt{2}x}{\sqrt{2}x+a}}\mathrm{d}x$$

$$= \sqrt{2}\int_0^{\frac{3a}{\sqrt{2}}} \frac{x}{\sqrt{3}}\cdot\sqrt{\frac{3a-\sqrt{2}x}{\sqrt{2}x+a}}\mathrm{d}\left(\sqrt{2}x\right)$$

$$= \sqrt{2}\int_0^{3a} \frac{t}{\sqrt{2}\sqrt{3}}\cdot\sqrt{\frac{3a-t}{t+a}}\mathrm{d}t\text{ , } t = \sqrt{2}x$$

$$= \frac{1}{\sqrt{3}}\int_0^{3a} t\cdot\sqrt{\frac{3a-t}{t+a}}\mathrm{d}t$$

Let $u = \sqrt{t+a}$; $t = 0$, $u = \sqrt{a}$; $t = 3a$, $u = 2\sqrt{a}$

$\mathrm{d}u = \dfrac{\mathrm{d}t}{2\sqrt{t+a}}$; $t = u^2 - a$

Area $= \dfrac{1}{\sqrt{3}}\displaystyle\int_{\sqrt{a}}^{2\sqrt{a}} \left(u^2-a\right)\cdot\sqrt{3a-u^2+a}\left(2\mathrm{d}u\right)$

$$= \frac{2}{\sqrt{3}}\int_{\sqrt{a}}^{2\sqrt{a}} \left(u^2-a\right)\cdot\sqrt{4a-u^2}\mathrm{d}u$$

Let $u = 2\sqrt{a}\sin\theta$, $\sqrt{4a-u^2} = 2\sqrt{a}\cos\theta$; $\mathrm{d}u = 2\sqrt{a}\cos\theta\mathrm{d}\theta$

$u = \sqrt{a}$, $\theta = \dfrac{\pi}{6}$; $u = 2\sqrt{a}$, $\theta = \dfrac{\pi}{2}$

Area $= \dfrac{2}{\sqrt{3}}\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(4a\sin^2\theta - a\right)\cdot\left(2\sqrt{a}\cos\theta\right)^2\mathrm{d}\theta$

$$= \frac{8a^2}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(4\sin^2\theta-1\right)\cdot\cos^2\theta\mathrm{d}\theta$$

$$= \frac{8a^2}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta\cos^2\theta\mathrm{d}\theta - \frac{8a^2}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2\theta\mathrm{d}\theta$$

$$= \frac{8a^2}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 2\theta d\theta - \frac{8a^2}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}\mathrm{d}\theta$$

$$= \frac{8a^2}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2}\mathrm{d}\theta - \frac{8a^2}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}\mathrm{d}\theta$$

$$= \frac{4a^2}{\sqrt{3}}\left(\theta-\frac{\sin 4\theta}{4}\right)_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{4a^2}{\sqrt{3}}\left(\theta+\frac{\sin 2\theta}{2}\right)_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{4a^2}{\sqrt{3}}\left(-\frac{\sin 4\theta}{4}-\frac{\sin 2\theta}{2}\right)_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{4a^2}{\sqrt{3}}\left(\frac{\sin\frac{2\pi}{3}}{4}+\frac{\sin\frac{\pi}{3}}{2}\right) = \frac{a^2}{\sqrt{3}}\left(\sin\frac{2\pi}{3}+2\sin\frac{\pi}{3}\right) = \frac{a^2}{\sqrt{3}}\left(\frac{3\sqrt{3}}{2}\right) = \frac{3a^2}{2}$$