# Mean Value Theorem

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Let f:[$a$, $b$] → **R** be continuous on [$a$, $b$] and differentiable on ($a$, $b$).

Then $\exists c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof: Consider the auxiliary function $g(x) = f(a) - f(x) + \frac{x - a}{b - a}[f(b) - f(a)]$

Then g($x$) is continuous on [$a$, $b$] and differentiable on ($a$, $b$).

$g(a) = g(b) = 0$ .

By Rolle's Theorem, $\exists c \in (a, b)$ s.t. $g'(c) = 0$ and hence $f'(c) = \frac{f(b) - f(a)}{b - a}$.

# Cauchy Mean Value Theorem

Let f and g be differentiable on ($a$, $b$) and continuous an [$a$, $b$],

then $\exists c \in (a, b)$ such that $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$ .

Proof: Define $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$

Then h is continuous on [$a$, $b$] and differentiable on ($a$, $b$) .

We have $h(a) = f(a)\,g(b) - g(a)f(b)$

$h(b) = f(a)g(b) - g(a)f(b)$

$\therefore h(a) = h(b)$

By Rolle’s Theorem, $\exists c \in (a, b)$ s.t. $h'(c) = 0$ and result follows.

**Example 1**

If $\frac{a_0}{n+1}+\frac{a_1}{n}+\cdots+\frac{a_{n-1}}{2}+a_n=0$,

prove that the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ has at least one root between 0 and 1.

Let $f(x)=\frac{a_0x^{n+1}}{n+1}+\frac{a_1x^n}{n}+\cdots+\frac{a_{n-1}x^2}{2}+a_nx$.

Then f($x$) is a polynomial which is continuously differentiable everywhere.

$f(0) = 0$ and $f(1)=\frac{a_0}{n+1}+\frac{a_1}{n}+\cdots+\frac{a_{n-1}}{2}+a_n=0$ (given)

By mean valued theorem, $\exists c \in (0, 1)$ such that $f'(c) = 0$

$f'(c) = a_0c^n + a_1c^{n-1} + \cdots + a_{n-1}c + a_n = 0$

i.e. the equation $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ has at least one root ($c$) between 0 and 1 .

**Example 2**

By using mean value theorem on f($x$) = cos $x$ (in radians) with $a=\frac{\pi}{3}+\frac{\pi}{180}$, $b=\frac{\pi}{3}$,

prove that $\frac{1}{2}-\frac{\pi}{360}>\cos 61^\circ>\frac{1}{2}-\frac{\pi}{180}$.

$f(x)=\cos x$, $a=\frac{\pi}{3}+\frac{\pi}{180}$, $b=\frac{\pi}{3}$

$$\exists\, c \in (b, a) \text{ such that } \frac{f(a)-f(b)}{a-b}=f'(c) \Rightarrow \frac{\cos\left(\frac{\pi}{3}+\frac{\pi}{180}\right)-\cos\left(\frac{\pi}{3}\right)}{\frac{\pi}{180}}=-\sin c$$

$$\frac{\pi}{180}\sin 30^\circ<\frac{\pi}{180}\sin c=\frac{1}{2}-\cos 61^\circ<\frac{\pi}{180}\sin 90^\circ$$

$$\frac{\pi}{360}<\frac{1}{2}-\cos 61^\circ<\frac{\pi}{180} \Rightarrow \frac{1}{2}-\frac{\pi}{360}>\cos 61^\circ>\frac{1}{2}-\frac{\pi}{180}$$

**Example 3**

(a) If f'($x$) exists in $0 < a \le x \le b$, show that there exists $c \in (a, b)$ such that

$$f(b) - f(a) = c\, f'(c) \ln\frac{b}{a}.$$

(Hint: Let $g(x) = [f(b) - f(a)] \ln\frac{x}{a} - f(x) \ln\frac{b}{a}$.)

(b) By taking $f(x) = x^{\frac{1}{n}}$, deduce that $\lim_{n\to\infty} n\left(a^{\frac{1}{n}} - 1\right) = \ln a$ for $a > 0$.

(a) Let $g(x) = [f(b) - f(a)] \ln\frac{x}{a} - f(x) \ln\frac{b}{a}$

$g(a) = -f(a) \ln\frac{b}{a}$; $g(b) = -f(a) \ln\frac{b}{a}$

By Rolle's Theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$

$$[f(b) - f(a)]\frac{1}{c} - f'(c) \ln\frac{b}{a} = 0$$

$$f(b) - f(a) = c\, f'(c) \ln\frac{b}{a}.$$

(b) When $a = 1$, LHS = RHS = 0

When $a > 1$, $n\left(a^{\frac{1}{n}} - 1\right) = n\, c\, f'(c) \ln a = c^{\frac{1}{n}} \ln a$, for some $c \in (1, a)$.

Let $c^{\frac{1}{n}} = 1 + h_n$, $h_n > 0$

Then $c = (1 + h_n)^n = 1 + nh_n + \cdots > 1 + nh_n$

$$\frac{c-1}{n} > h_n > 0$$

$$\lim_{n\to\infty}\frac{c-1}{n} \ge \lim_{n\to\infty} h_n \ge \lim_{n\to\infty} 0$$

By squeezing principle, $\lim_{n\to\infty} h_n = 0$

$$\therefore \lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} n\left(a^{\frac{1}{n}} - 1\right) = \ln a$$

When $0 < a < 1$, $n\left(a^{\frac{1}{n}} - 1\right) = c^{\frac{1}{n}} \ln a$, for some $c \in (a, 1)$.

$$\frac{1}{c^{\frac{1}{n}}} > 1 \Rightarrow \lim_{n\to\infty}\frac{1}{c^{\frac{1}{n}}} = 1 \Rightarrow \lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

$$\therefore \lim_{n\to\infty} n\left(a^{\frac{1}{n}} - 1\right) = \ln a$$

# Mean Value Theorem for Integrals

Calculus Volume 2 Second Edition by Tom M.APOSTOL p.154, 219

**Theorem 1** If f is continuous on [$a$, $b$], then for some $c$ in [$a$,$b$] we have $\int_a^b f(x)\,dx = f(c)(b-a)$

Proof: Let Max(f($x$)) = M, Min(f($x$)) = $m$ .

$m \le f(x) \le M$

$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$

By the intermediate-value theorem, there exists a constant $c$: $a \le c \le b$, such that

$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$

$\int_a^b f(x)\,dx = f(c)(b-a)$

**Theorem 2 Weighted Mean-Valued Theorem for Integrals**

Assume $f$ and $g$ are continuous on [$a$, $b$] . If $g$ is never changes sign in [$a$,$b$], then there exists $c \in [a,b]$ such that $\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$.

Proof: If g($x$) ≥ 0 for all $x \in [a,b]$, then $\int_a^b g(x)\,dx \ge 0$

Let Max(f($x$)) = $M$, Min(f($x$)) = $m$.

$mg(x) \le f(x)\,g(x) \le Mg(x)$

$m\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx$ ………… (*)

If $\int_a^b g(x)\,dx = 0$, then g($x$) = 0 for all $x \in [a, b]$.

(otherwise $\exists x_0 \in (a,b)$ such that $g(x_0) > 0$,

since g is continuous, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$

$-\varepsilon < g(x) - g(x_0) < \varepsilon$

$-\varepsilon + g(x_0) < g(x) < \varepsilon + g(x_0)$

let $\varepsilon = \frac{1}{2}g(x_0) > 0$, then $\frac{1}{2}g(x_0) < g(x) < \frac{3}{2}g(x_0)$, for all $x$: $|x - x_0| < \delta$

$\int_a^b g(x)\,dx \ \ge \ \int_{x_0-\delta}^{x_0+\delta} g(x)\,dx > 0$, which leads to a contradiction.)

In this case, $\int_a^b f(x)g(x)\,dx = \int_a^b 0\,dx = 0 = f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx$, $c = \frac{a+b}{2}$

If $\int_a^b g(x)\,dx \ne 0$, then $\int_a^b g(x)\,dx > 0$; divide (*) by $\int_a^b g(x)\,dx$.

$$m \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le M$$

By the intermediate-value theorem, there exists a constant $c$: $a < c < b$, such that

$$\frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} = f(c)$$

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

If $g(x) \le 0$, we can arrive at the same result if we apply on $-g(x) \ge 0$ .

**Theorem 3 Second Mean Value Theorem for Integrals**

Assume $g$ is continuous on $[a, b]$, and assume f has a derivative which is continuous and never change sign in $[a, b]$ . Then there exists $c\in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx$$

**Proof:** Let $G(x) = \int_a^x g(t)\,dt$, since g is continuous, we have $G'(x) = g(x)$

Therefore integrating by parts gives

$$\int_a^b f(x)g(x)\,dx = \int_a^b f(x)G'(x)\,dx = \int_a^b f(x)\,dG(x)$$

$$= f(b)G(b) - \int_a^b f'(x)G(x)\,dx \quad (\because G(a) = 0)$$

By Theorem 2, we have $\int_a^b f'(x)G(x)\,dx = G(c)\int_a^b f'(x)\,dx$ for some $c\in [a, b]$ .

$$= G(c)[f(b) - f(a)]$$

$$\int_a^b f(x)g(x)\,dx = f(b)G(b) - G(c)[f(b) - f(a)]$$

$$= f(b)\int_a^b g(t)\,dt - [f(b) - f(a)]\int_a^c g(t)\,dt$$

$$= f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx$$

**HKAL Past Paper 1995 Paper 2 Q13(b)**

Let F($x$) be a function with a continuous second derivative such that $F''(x) \geq 0$ and $F'(x) \geq m > 0$ for $a \leq x \leq b$. Using Theorem 3 with $f(x) = -\frac{1}{F'(x)}$ and $g(x) = -F'(x)\cos F(x)$,

show that $\left|\int_a^b \cos F(x)\,dx\right| \leq \frac{4}{m}$.

$f'(x) = \frac{F''(x)}{\left[F'(x)\right]^2} > 0$ for all $x \in [a, b]$, so f($x$) satisfies the conditions in Theorem 3 .

$$\int_a^b \cos F(x)\,dx = f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx, \text{ for some } c \in [a,b]$$

$$= -\frac{1}{F'(a)}\int_a^c -F'(x)\cos F(x)\,dx - \frac{1}{F'(b)}\int_c^b -F'(x)\cos F(x)\,dx$$

$$= \frac{1}{F'(a)}\int_a^c F'(x)\cos F(x)\,dx + \frac{1}{F'(b)}\int_c^b F'(x)\cos F(x)\,dx$$

$$= \frac{1}{F'(a)}\int_a^c \cos F(x)\,dF(x) + \frac{1}{F'(b)}\int_c^b \cos F(x)\,dF(x)$$

$$= \frac{1}{F'(a)}\sin F(x)\Big|_a^c + \frac{1}{F'(b)}\sin F(x)\Big|_c^b$$

$$\left|\int_a^b \cos F(x)\,dx\right| = \left|\frac{1}{F'(a)}\left[\sin F(c) - \sin F(a)\right] + \frac{1}{F'(b)}\left[\sin F(b) - \sin F(c)\right]\right|$$

$$\leq \frac{\left|\sin F(c)\right|}{\left|F'(a)\right|} + \frac{\left|\sin F(a)\right|}{\left|F'(a)\right|} + \frac{\left|\sin F(b)\right|}{\left|F'(b)\right|} + \frac{\left|\sin F(c)\right|}{\left|F'(b)\right|}$$

$$\leq \frac{4}{m}$$

(c) (i) Show that $\int_0^1 \cos(x^n)\,dx \leq \int_0^1 \cos(x^{n+1})\,dx$

Hence show that $\lim_{n\to\infty}\int_0^1 \cos(x^n)\,dx$ exists.

(ii) Using (b), or otherwise, show that $\lim_{n\to\infty}\int_0^{2\pi} \cos(x^n)\,dx$ exists.

(i) For $0 \leq x \leq 1$, $x^n \geq x^{n+1}$,

$0 \leq \cos(x^n) \leq \cos(x^{n+1})$

$\int_0^1 \cos(x^n)\,dx \leq \int_0^1 \cos(x^{n+1})\,dx$

The sequence $\left\{\int_0^1 \cos(x^n)\,dx\right\}$ is monotonic increasing which is bounded above by 1 .

By Monotonic convergent Theorem, $\lim_{n\to\infty}\int_0^1 \cos(x^n)\,dx$ exists.

(ii) Let $F(x) = x^n$ ; $1 \le x \le 2\pi$ .

For $n \ge 2$, $F'(x) = nx^{n-1} \ge n > 0$ and $F''(x) = n(n-1)x^{n-2} > 0$

$\therefore$ By (b), $\left|\int_1^{2\pi} \cos x^n dx\right| \le \frac{4}{n}$

$\Rightarrow \lim_{n\to\infty}\left|\int_1^{2\pi} \cos x^n \mathrm{d}x\right| = 0$

$\Rightarrow \lim_{n\to\infty}\int_1^{2\pi} \cos x^n \mathrm{d}x = 0$

$$\lim_{n\to\infty}\int_0^{2\pi} \cos\left(x^n\right)\mathrm{d}x = \lim_{n\to\infty}\int_0^{1} \cos\left(x^n\right)\mathrm{d}x + \lim_{n\to\infty}\int_1^{2\pi} \cos\left(x^n\right)\mathrm{d}x$$

$$= \lim_{n\to\infty}\int_0^{1} \cos\left(x^n\right)\mathrm{d}x$$

$\therefore$ The limit exists.