Taylor's Theorem

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Theorem 1 (Lagrange form)

Suppose $f^{(n-1)}(x)$ is continuous on $[a, b]$ and differentiable on (a, b),

then $\forall x \in (a, b)\ \exists\ c \in (a, x)$ such that $f(x) = f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!}(x-a)^r + \frac{f^{(n)}(c)}{n!}(x-a)^n$

Proof: Let $g(t) = (t - a)^n$; Note: $g^{(r)}(a) = 0$ for $0 \le r < n$ and $g^{(n)}(t) = n!$

Define $F(t) = f(t) + \sum_{r=1}^{n-1} \frac{f^{(r)}(t)}{r!}(x-t)^r$; $G(t) = g(t) + \sum_{r=1}^{n-1} \frac{g^{(r)}(t)}{r!}(x-t)^r$.

F, G satisfies the conditions of Cauchy's mean value theorem

then $\exists\ c \in (a, x)$ such that $[F(x) - F(a)]G'(c) = [G(x) - G(a)]F'(c)$

$F(t) = f(t) + f'(t)(x - t) + \frac{f^{(2)}(t)}{2!}(x-t)^2 + \cdots + \frac{f^{(n-1)}(t)}{(n-1)!}(x-t)^{n-1}$

$$F'(t) = f'(t) - f'(t) + f''(t)(x-t) - f''(t)(x-t) + \frac{f^{(3)}(t)}{2!}(x-t)^2 - \frac{f^{(3)}(t)}{2!}(x-t)^2 + \cdots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}$$

$$= \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}$$

$$G'(t) = \frac{g^{(n)}(t)}{(n-1)!}(x-t)^{n-1} = \frac{n!}{(n-1)!}(x-t)^{n-1} = n(x-t)^{n-1}$$

$$\left[f(x) - f(a) - \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!}(x-a)^r\right] n(x-c)^{n-1} = \left[(x-a)^n - 0\right] \frac{f^{(n)}(c)}{(n-1)!}(x-c)^{n-1}$$

$$f(x) = f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!}(x-a)^r + \frac{f^{(n)}(c)}{n!}(x-a)^n \text{ . Q.E.D.}$$

$f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!}(x-a)^r$ is called the **Taylor Polynomial of degree $n - 1$**.

$\frac{f^{(n)}(c)}{n!}(x-a)^n$ is **the remainder of Lagrange form**

Theorem 2 (Cauchy form)

Suppose $f^{(n-1)}(x)$ is continuous on $[a, b]$ and differentiable on (a, b),

then $\forall x \in (a, b)\ \exists\, c \in (a, x)$ such that $f(x) = f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!}(x-a)^r + \frac{f^{(n)}(c)}{(n-1)!}(x-a)(x-c)^{n-1}$

Proof: Define $F(t) = f(x) - f(t) - \sum_{r=1}^{n-1} \frac{f^{(r)}(t)}{r!}(x-t)^r$

$$F'(t) = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}$$

By mean value theorem, $\exists\, c \in (a, x)$ such that $\frac{F(x)-F(a)}{x-a} = F'(c)$

$$\frac{0 - f(x) + f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!}(x-a)^r}{x-a} = -\frac{f^{(n)}(c)}{(n-1)!}(x-c)^{n-1}$$

$f(x) = f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!}(x-a)^r + \frac{f^{(n)}(c)}{(n-1)!}(x-a)(x-c)^{n-1}$. Q.E.D.

Corollary (Maclaurin's Theorem)

Put $a = 0$, then $\forall x \in (a, b)\ \exists\, c \in (a, x)$ such that $f(x) = f(0) + \sum_{r=1}^{n-1} \frac{f^{(r)}(0)}{r!} x^r + \frac{f^{(n)}(c)}{n!} x^n$.

Examples

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n-1}}{(2n-1)!} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \cdots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \cdots + \frac{x^n}{n!} + \cdots$

$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n \frac{x^n}{n} + \cdots$; $|x| < 1$ ($\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n} + \cdots$)

$-\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots$; $|x| < 1$

$\ln\left(\frac{1-x}{1+x}\right) = 2\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots\right)$; $|x| < 1$

Theorem 3 (Integral form) 1984 Paper 2 Q2, 2003 Paper 2 Q12

If f is n times continuously differentiable on $[a-h, a+h]$,then

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^k+\int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

Proof: Let $I_m = \frac{1}{(m-1)!}\int_0^x (x-t)^{m-1}f^{(m)}(t)\,dt$, $m\ge 1$

$$I_{m+1}=\frac{1}{m!}\int_0^x (x-t)^m f^{(m+1)}(t)\,dt=\frac{1}{m!}\int_0^x (x-t)^m\,df^{(m)}(t)$$

$$=\frac{1}{m!}\left[(x-t)^m f^{(m)}(t)\right]_0^x-\frac{1}{m!}\int_0^x f^{(m)}(t)\,d(x-t)^m \quad \text{(integration by parts)}$$

$$=-\frac{1}{m!}x^m f^{(m)}(0)+\frac{1}{(m-1)!}\int_0^x (x-t)^{m-1}f^{(m)}(t)\,dt$$

$$=I_m-\frac{f^{(m)}(0)}{m!}x^m$$

$$I_n=I_{n-1}-\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

$$=I_{n-2}-\frac{f^{(n-2)}(0)}{(n-2)!}x^{n-2}-\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

$$=\cdots\cdots$$

$$=I_1-\frac{f^{(1)}(0)}{1!}x-\cdots\cdots-\frac{f^{(n-2)}(0)}{(n-2)!}x^{n-2}-\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

On the other hand, $I_1=\frac{1}{1!}\int_0^x (x-t)^{1-1}f^{(1)}(t)\,dt=\int_0^x f'(t)\,dt=\int_0^x df(t) = f(x)-f(0)$

$$\therefore I_n=f(x)-f(0)-\frac{f^{(1)}(0)}{1!}x-\ldots\ldots-\frac{f^{(n-2)}(0)}{(n-2)!}x^{n-2}-\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}$$

$$f(x)=f(0)+\frac{f^{(1)}(0)}{1!}x+\cdots\cdots+\frac{f^{(n-2)}(0)}{(n-2)!}x^{n-2}+\frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}+\frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,dt$$

1984 Paper 2 Q2*c* Show that $0<\ln(1+x)-x+\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{4}x^4<\frac{1}{5}x^5$, for $0<x<1$

Putting $f(x) = 1n(1+x)$ which is infinitely differentiable on $(-1, 1)$,

$f'(x)=\frac{1}{1+x}$; $f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

For any $0<x<1$, $1n(1+x) = 1n\ 1 + x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+I_5$

Since $I_5=\frac{1}{4!}\int_0^x (x-t)^4\cdot\frac{(-1)^4(4!)}{(1+t)^5}dt$

> 0 as $(x-t)^4$, $(1+x)^5>0$ for $t\in(0, x)$

$\therefore \text{In}\,(1+x)-x+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{3}=I_5>0$

Similarly, $I_6 = \frac{1}{5!}\int_0^x \frac{(x-t)^5(-1)^5(5!)}{(1+t)^6}\mathrm{d}t < 0$

$\therefore \ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x}{4} = I_6 + \frac{x^5}{5} < \frac{x^5}{5}$

2003 Paper 2Q12 (b)

Define $g(x) = \frac{1}{\sqrt{1-x^2}}$ for all $x \in (-1, 1)$. Let n be a positive integer.

(i) Prove that $(1 - x^2)g'(x) - x\,g(x) = 0$.
Hence deduce that $(1 - x^2)\,g^{(n+1)}(x) - (2n + 1)\,x\,g^{(n)}(x) - n^2\,g^{(n-1)}(x) = 0$, where $g^{(0)} = g$.

(ii) Prove that $g^{(2n-1)}(0) = 0$ and $g^{(2n)}(0) = \left(\frac{(2n)!}{(2^n)(n!)}\right)^2$.

(iii) Using (a), prove that $g(x) = \sum_{k=0}^{n-1}\frac{C_k^{2n}}{2^{2k}}x^{2k} + \frac{1}{(2n-1)!}\int_0^x (x-t)^{2n-1} g^{(2n)}(t)\,\mathrm{d}t$

(i) $(1 - x^2)g^2(x) = 1$
Differentiate both sides w.r.t. x :
$2(1 - x^2)g(x)g'(x) - 2x\,g^2(x) = 0$
$\because g(x) \neq 0,\ (1 - x^2)g'(x) - x\,g(x) = 0$ ·········(*)
Use Leibniz rule to differentiate (*) w.r.t. x n times.
$(1 - x^2)\,g^{(n+1)}(x) - 2nx\,g^{(n)}(x) - n(n - 1)g^{(n-1)}(x) - x\,g^{(n)}(x) - n\,g^{(n-1)}(x) = 0$
$(1 - x^2)\,g^{(n+1)}(x) - (2n + 1)\,x\,g^{(n)}(x) - n^2\,g^{(n-1)}(x) = 0$ ········· (**)

(ii) $g(0) = 1 = \left(\frac{(2\times 0)!}{(2^0)(0!)}\right)^2$
Put x= 0 in (*), $g'(0) = 0$
Put $x = 0$, $n = 1$ into (**), $g''(0) = 1 = \left(\frac{2!}{(2^1)(1!)}\right)^2$
$\therefore$ the statement is true for $n = 1$
Suppose $g^{(2k-1)}(0) = 0$ and $g^{(2k)}(0) = \left(\frac{(2k)!}{(2^k)(k!)}\right)^2$
Put $x = 0$, $n = 2k$ into (**): $g^{(2k+1)}(0) = (2k)^2\,g^{2k-1}(0) = 0$
Put $x = 0$, $n = 2k$+1 into (**): $g^{(2k+2)}(0) = (2k + 1)^2\,g^{(2n)}(0) = \left(\frac{(2k+1)!}{(2^k)(k!)}\right)^2 = \left[\frac{(2k+2)!}{(2^{k+1})(k+1)!}\right]^2$
The statement is also true for $n = k + 1$, by M. I., the statement is true for all $n \in \mathbf{N} \cup \{0\}$

(iii) **By Taylor's Theorem (integral form),** (replace n by $2n$, put $a = 0$)

$$g(x) = \sum_{k=0}^{2n-1}\frac{g^{(k)}(0)}{k!}x^k + \int_0^x g^{(2n)}(t)\frac{(x-t)^{2n-1}}{(2n-1)!}\mathrm{d}t$$

$$= \sum_{k=0}^{n-1}\frac{\left[\frac{(2k)\ !}{(2^k)(k\ !)}\right]^2}{(2k)\ !}x^{2k} + \frac{1}{(2n-1)\ !}\int_0^x (x-t)^{2n-1} g^{(2n)}(t)\,\mathrm{d}t$$

$$= \sum_{k=0}^{n-1}\frac{C_k^{2n}}{2^{2k}}x^{2k} + \frac{1}{(2n-1)\ !}\int_0^x (x-t)^{2n-1} g^{(2n)}(t)\,\mathrm{d}t$$