Hyperbolic Functions

First created by Mr. Francis Hung in 1989, retyped as MS WORD document on 20080609. Last updated: 13 February 2022

Reference: Techniques of Mathematical Analysis by C.T. Tranter p. 160 - p.165, p.263 - p.266

Euler's Function: $e^{ix} = \cos x + i \sin x$; $e^{-ix} = \cos x - i \sin x$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix});\ \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$e^{x+iy} = e^{x}(\cos y + i \sin y)$$

Hyperbolic Function: $\cosh z = \frac{1}{2}(e^{z} + e^{-z})$ $\qquad$ $\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^{z} + e^{-z}}$

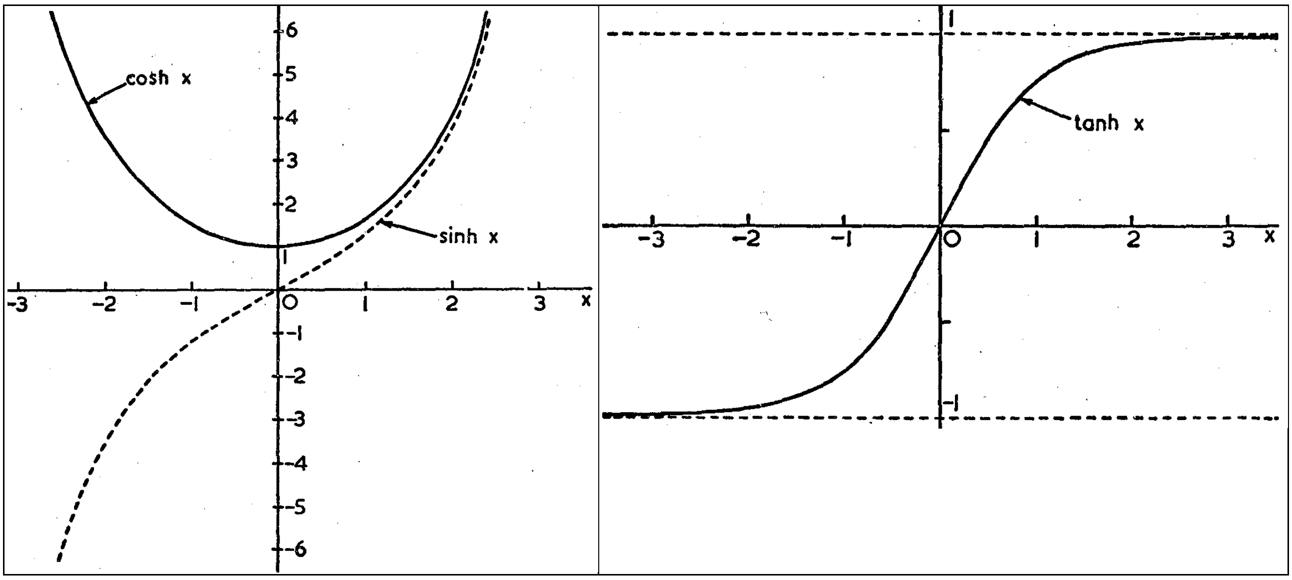
$$\sinh z = \frac{1}{2}(e^{z} - e^{-z}) \qquad \operatorname{cosech} z = \frac{1}{\sinh z} = \frac{2}{e^{z} - e^{-z}}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} \qquad \coth z = \frac{1}{\tanh z} = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$$

$$\cosh z + \sinh z = e^{z} \qquad \cosh z - \sinh z = e^{-z}$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots; \qquad \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n-1}}{(2n-1)!} + \cdots$$

Graphs



Relation with real trigonometric functions:

$\cosh z = \cos iz$; $\qquad$ $\sinh z = -i \sin iz$

$\tanh z = -i \tan iz$; $\qquad$ $\coth z = i \cot iz$

$\operatorname{sech} z = \sec iz$; $\qquad$ $\operatorname{cosech} z = i \operatorname{cosec} iz$

Identities: $\cosh^2 z - \sinh^2 z = 1$

$\operatorname{sech}^2 z = 1 - \tanh^2 z$

$\operatorname{cosech}^2 z = -1 + \coth^2 z$

Exercise: Find $\sinh(-z)$; $\cosh(-z)$; $\sinh 0$; $\cosh 0$.

Addition Formulae: $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

Double Angle Formulae: $\sinh 2z = 2 \sinh z \cosh z$

$\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 2\sinh^2 z + 1$

Special manipulation: If x and y are real numbers,

$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$
$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$
$\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$
$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$

Inverse hyperbolic function: z is a complex number, $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ for $-1 < z < 1$

$\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$ $\quad$ $\operatorname{cosech}^{-1} x = \ln\left(\frac{1\pm\sqrt{1+x^2}}{x}\right)$, $x \neq 0$
($x > 0$, take +; $x < 0$, take –)

$\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right)$, $x \geq 1$ $\quad$ $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, $0 < x \leq 1$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$ $\quad$ $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $|x| > 1$

Differentiation:

$\frac{\mathrm{d}\sinh x}{\mathrm{d}x} = \cosh x$ $\quad$ $\frac{\mathrm{d}\operatorname{cosech} x}{\mathrm{d}x} = -\operatorname{cosech} x \coth x$

$\frac{\mathrm{d}\cosh x}{\mathrm{d}x} = \sinh x$ $\quad$ $\frac{\mathrm{d}\operatorname{sech} x}{\mathrm{d}x} = -\operatorname{sech} x \tanh x$

$\frac{\mathrm{d}\tanh x}{\mathrm{d}x} = \operatorname{sech}^2 x$ $\quad$ $\frac{\mathrm{d}\coth x}{\mathrm{d}x} = -\operatorname{cosech}^2 x$

$\frac{\mathrm{d}\sinh^{-1} x}{\mathrm{d}x} = \frac{1}{\sqrt{1+x^2}}$ $\quad$ $\frac{\mathrm{d}\cosh^{-1} x}{\mathrm{d}x} = \frac{1}{\sqrt{x^2-1}}$, for all $x > 1$

$\frac{\mathrm{d}\tanh^{-1} x}{\mathrm{d}x} = \frac{1}{1-x^2}$, for $|x| < 1$ $\quad$ $\frac{\mathrm{d}\coth^{-1} x}{\mathrm{d}x} = \frac{1}{1-x^2}$, $|x| > 1$

$\frac{\mathrm{d}\operatorname{sech}^{-1} x}{\mathrm{d}x} = -\frac{1}{x\sqrt{1-x^2}}$, $0 < x < 1$ $\quad$ $\frac{\mathrm{d}\operatorname{cosech}^{-1} x}{\mathrm{d}x} = \frac{-1}{|x|\sqrt{1+x^2}}$, for $x \neq 0$

Integration: The inverse process of differentiation

$\int \sinh x\,\mathrm{d}x = \cosh x + c$ $\quad$ $\int \operatorname{cosech} x \coth x\,\mathrm{d}x = -\operatorname{cosech} x + c$

$\int \cosh x\,\mathrm{d}x = \sinh x + c$ $\quad$ $\int \operatorname{sech} x \tanh x\,\mathrm{d}x = -\operatorname{sech} x + c$

$\int \operatorname{sech}^2 x\,\mathrm{d}x = \tanh x + c$ $\quad$ $\int \operatorname{cosech}^2 x\,\mathrm{d}x = -\coth x + c$

$$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2+1}\right| + c = \sinh^{-1} x + c$$

$$\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = \ln\left|x+\sqrt{x^2-1}\right| + c = \cosh^{-1} x + c$$

$$\int \frac{1}{1-x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c = \tanh^{-1} x + c$$

$$\int \frac{1}{x^2-1}\,\mathrm{d}x = -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + c = -\coth^{-1} x + c$$

$$\int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x = -\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + c = -\operatorname{sech}^{-1} x + c$$

$$\int \frac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x = -\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right| + c = -\operatorname{cosech}^{-1} x + c$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots$$

$$\begin{aligned}\cosh z &= \frac{1}{2}(e^z + e^{-z}) \\ &= \frac{1}{2}\left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots\right) + \frac{1}{2}\left(1 - z + \frac{z^2}{2!} - \cdots + (-1)^n \cdot \frac{z^n}{n!} + \cdots\right) \\ &= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots\end{aligned}$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n-1}}{(2n-1)!} + \cdots$$

$$\begin{aligned}\sinh z &= \frac{1}{2}(e^z - e^{-z}) \\ &= \frac{1}{2}\left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots\right) - \frac{1}{2}\left(1 - z + \frac{z^2}{2!} - \cdots + (-1)^n \cdot \frac{z^n}{n!} + \cdots\right) \\ &= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n-1}}{(2n-1)!} + \cdots\end{aligned}$$

$\cosh z = \cos iz$ $\cos iz = \frac{1}{2}(e^{i\cdot iz} + e^{-i\cdot iz})$ $= \frac{1}{2}(e^{-z} + e^{z})$ $= \cosh z$	$\sinh z = -i \sin iz$ $-i \sin iz = -i \cdot \frac{1}{2i}(e^{i\cdot iz} - e^{-i\cdot iz})$ $= -\frac{1}{2}(e^{-z} - e^{z})$ $= \sinh z$
$\tanh z = -i \tan iz$ $-i \tan iz = -i \cdot \dfrac{\frac{1}{2i}(e^{i\cdot iz} - e^{-i\cdot iz})}{\frac{1}{2}(e^{i\cdot iz} + e^{-i\cdot iz})}$ $= -\dfrac{(e^{-z} - e^{z})}{(e^{-z} + e^{z})}$ $= \tanh z$	$\coth z = i \cot iz$ $i \cot iz = i \cdot \dfrac{\frac{1}{2}(e^{i\cdot iz} + e^{-i\cdot iz})}{\frac{1}{2i}(e^{i\cdot iz} - e^{-i\cdot iz})}$ $= -\dfrac{(e^{-z} + e^{z})}{(e^{-z} - e^{z})}$ $= \coth z$
$\operatorname{sech} z = \sec iz$ $\sec iz = \dfrac{1}{\cos iz}$ $= \dfrac{2}{e^{i\cdot iz} + e^{-i\cdot iz}}$ $= \dfrac{2}{e^{-z} + e^{z}}$ $= \operatorname{sech} z$	$\operatorname{cosech} z = i \operatorname{cosec} iz$ $i \operatorname{cosec} iz = i \cdot \dfrac{1}{\sin iz}$ $= i \cdot \dfrac{1}{\frac{1}{2i}(e^{i\cdot iz} - e^{-i\cdot iz})}$ $= -\dfrac{2}{(e^{-z} - e^{z})} = \operatorname{cosech} z$
$\cosh^2 z - \sinh^2 z = 1$ L.H.S. $= \frac{1}{4}(e^z + e^{-z})^2 - \frac{1}{4}(e^z - e^{-z})^2$ $= \frac{1}{4}[(e^z + e^{-z})^2 - (e^z - e^{-z})^2]$ $= 1 =$ R.H.S.	$\operatorname{sech}^2 z = 1 - \tanh^2 z$ $1 - \tanh^2 z = \left(1 + \dfrac{e^z - e^{-z}}{e^z + e^{-z}}\right)\left(1 - \dfrac{e^z - e^{-z}}{e^z + e^{-z}}\right)$ $= \dfrac{2e^z}{e^z + e^{-z}} \cdot \dfrac{2e^{-z}}{e^z + e^{-z}} = \left(\dfrac{2}{e^z + e^{-z}}\right)^2$ $= \operatorname{sech}^2 z$

$\text{cosech}^2 z = -1 + \coth^2 z$ $-1 + \coth^2 z = \left(\frac{e^z + e^{-z}}{e^z - e^{-z}} + 1\right)\left(\frac{e^z + e^{-z}}{e^z - e^{-z}} - 1\right)$ $= \frac{2e^z}{e^z - e^{-z}} \cdot \frac{2e^{-z}}{e^z - e^{-z}} = \left(\frac{2}{e^z - e^{-z}}\right)^2$ $= \text{cosech}^2 z$	$\sinh(-z) = \frac{1}{2}(e^{-z} - e^{z}) = -\frac{1}{2}(e^{z} - e^{-z}) = -\sinh z$ $\cosh(-z) = \frac{1}{2}(e^{-z} + e^{z}) = \cosh z$ $\sinh 0 = \frac{1}{2}(e^0 - e^{-0}) = 0$; $\cosh 0 = \frac{1}{2}(e^0 + e^{-0}) = 1$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\sinh x \cosh y + \cosh x \sinh y$

$= \frac{1}{2}(e^x - e^{-x}) \cdot \frac{1}{2}(e^y + e^{-y}) + \frac{1}{2}(e^x + e^{-x}) \cdot \frac{1}{2}(e^y - e^{-y})$

$= \frac{1}{4}(e^x - e^{-x})(e^y + e^{-y}) + \frac{1}{4}(e^x + e^{-x})(e^y - e^{-y})$

$= \frac{1}{4}(e^{x+y} - e^{-x+y} + e^{x-y} - e^{-x-y}) + \frac{1}{4}(e^{x+y} + e^{-x+y} - e^{x-y} - e^{-x-y})$

$= \frac{1}{2}(e^{x+y} - e^{-x-y})$

$= \sinh(x + y)$

The proof of $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$ is similar and omitted.

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$\cosh x \cosh y + \sinh x \sinh y$

$= \frac{1}{2}(e^x + e^{-x}) \cdot \frac{1}{2}(e^y + e^{-y}) + \frac{1}{2}(e^x - e^{-x}) \cdot \frac{1}{2}(e^y - e^{-y})$

$= \frac{1}{4}(e^x + e^{-x})(e^y + e^{-y}) + \frac{1}{4}(e^x - e^{-x})(e^y - e^{-y})$

$= \frac{1}{4}(e^{x+y} + e^{-x+y} + e^{x-y} + e^{-x-y}) + \frac{1}{4}(e^{x+y} - e^{-x+y} - e^{x-y} + e^{-x-y})$

$= \frac{1}{2}(e^{x+y} + e^{-x-y})$

$= \cosh(x + y)$

The proof of $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$ is similar and omitted.

$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$ $\tanh(x + y) = \frac{\sinh(x+y)}{\cosh(x+y)}$ $= \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}$ $= \frac{\frac{\sinh x \cosh y}{\cosh x \cosh y} + \frac{\cosh x \sinh y}{\cosh x \cosh y}}{\frac{\cosh x \cosh y}{\cosh x \cosh y} + \frac{\sinh x \sinh y}{\cosh x \cosh y}}$ $= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$ The proof of $\tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$ is similar and omitted.	$\sinh 2z = 2 \sinh z \cosh z$ $\sinh 2z = \sinh(z + z)$ $= \sinh z \cosh z + \cosh z \sinh z$ $= 2 \sinh z \cosh z$ $\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 2\sinh^2 z + 1$ $\cosh 2z = \cosh(z + z)$ $= \cosh z \cosh z + \sinh z \sinh z$ $= \cosh^2 z + \sinh^2 z$ …… (*) $= \cosh^2 z + \cosh^2 z - 1$, using $\cosh^2 z - \sinh^2 z = 1$ $= 2\cosh^2 z - 1$ …… (**) $= 1 + \sinh^2 z + \sinh^2 z$, sub. the identity into (*) $= 2\sinh^2 z + 1$ …… (***)

$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

$\sin(x + iy) = \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right) = \frac{1}{2i}\left(e^{-y+ix} - e^{y-ix}\right)$

$\sin x \cosh y + i \cos x \sinh y$

$= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\cdot\frac{1}{2}\left(e^{y} + e^{-y}\right) + i\cdot\frac{1}{2}\left(e^{x} + e^{-x}\right)\cdot\frac{1}{2}\left(e^{y} - e^{-y}\right)$

$= \frac{1}{4i}\left(e^{y+ix} - e^{y-ix} + e^{-y+ix} - e^{-y-ix}\right) - \frac{1}{4i}\left(e^{y+ix} + e^{y-ix} - e^{-y+ix} - e^{-y-ix}\right)$

$= \frac{2}{4i}\left(-e^{y-ix} + e^{-y+ix}\right)$

$= \frac{1}{2i}\left(e^{-y+ix} - e^{y-ix}\right) = \sin(x + iy)$

$\cos(x + iy) = \cos x \cosh y - i \cos x \sinh y$

$\cos(x + iy) = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) = \frac{1}{2}\left(e^{-y+ix} + e^{y-ix}\right)$

$\cos x \cosh y - i \sin x \sinh y$

$= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)\cdot\frac{1}{2}\left(e^{y} + e^{-y}\right) - i\cdot\frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\cdot\frac{1}{2}\left(e^{y} - e^{-y}\right)$

$= \frac{1}{4}\left(e^{y+ix} + e^{y-ix} + e^{-y+ix} + e^{-y-ix}\right) - \frac{1}{4}\left(e^{y+ix} - e^{y-ix} - e^{-y+ix} + e^{-y-ix}\right)$

$= \frac{2}{4}\left(e^{y-ix} + e^{-y+ix}\right)$

$= \frac{1}{2}\left(e^{y-ix} + e^{-y+ix}\right) = \cos(x + iy)$

$\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$ $\sinh(x + iy) = \sinh x \cosh iy + \cosh x \sinh iy$ $= \sinh x \cos y + \cosh x \cdot \frac{1}{2}\left(e^{iy} - e^{iy}\right)$ $= \sinh x \cos y + i \cosh x \cdot \frac{1}{2i}\left(e^{iy} - e^{iy}\right)$ $= \sinh x \cos y + i \cosh x \sin y$	$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$ $\cosh(x + iy) = \cosh x \cosh iy + \sinh x \sinh iy$ $= \cosh x \cos y + \sinh x \cdot \frac{1}{2}\left(e^{iy} - e^{iy}\right)$ $= \sinh x \cos y + i \sinh x \cdot \frac{1}{2i}\left(e^{iy} - e^{iy}\right)$ $= \sinh x \cos y + i \sinh x \sin y$
$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ for $-1 < z < 1$ Let $y = \tanh^{-1} z$, then $z = \tanh y$ $z = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}$ $(e^{2y} + 1)z = e^{2y} - 1$ $e^{2y}\cdot(1 - z) = 1 + z$ $e^{2y} = \frac{1+z}{1-z}$ $2y = \log\left(\frac{1+z}{1-z}\right)$ $y = \tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ for $-1 < z < 1$	$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ Let $y = \sinh^{-1} x$, then $\sinh y = x$ $\frac{1}{2}(e^{y} - e^{-y}) = x$ $e^{2y} - 2x\, e^{y} - 1 = 0$ $e^{y} = x + \sqrt{x^2+1}$ or $x - \sqrt{x^2+1}$ $\because$ $e^{y} > 0$ and $x - \sqrt{x^2+1} < x - x = 0$ $\therefore$ $e^{y} = x + \sqrt{x^2+1}$ only $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

$\cosh^{-1} x = \ln\left(x+\sqrt{x^2-1}\right)$, for $x \ge 1$ Let $y = \cosh^{-1} x$, then $x = \cosh y$ $x = \frac{1}{2}(e^y + e^{-y})$, y is well defined only when $x > 0$ $e^{2y} - 2x\, e^y + 1 = 0$ $e^y = x+\sqrt{x^2-1}$ or $x-\sqrt{x^2-1}$ $\because (x+\sqrt{x^2-1})(x-\sqrt{x^2-1}) = 1$ $\therefore\ x-\sqrt{x^2-1} = \frac{1}{x+\sqrt{x^2-1}} = \left(x+\sqrt{x^2-1}\right)^{-1}$ $e^y = x+\sqrt{x^2-1}$ or $\left(x+\sqrt{x^2-1}\right)^{-1}$ $y = \ln(x+\sqrt{x^2-1})$ or $\ln\left(x+\sqrt{x^2-1}\right)^{-1}$ $y = \ln(x+\sqrt{x^2-1})$ or $-\ln(x+\sqrt{x^2-1})$ y is well defined only when $x \ge 1$ or $x \le -1$ It is convenient to adopt the positive sign and to write $y = \cosh^{-1} x = \ln(x+\sqrt{x^2-1})$	$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$ Let $y = \tanh^{-1} x$, then $x = \tanh y$ $x = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{e^{2z}-1}{e^{2z}+1}$ $xe^{2z} + x = e^{2z} - 1$ $x + 1 = e^{2z}(1 - x)$ $e^{2z} = \frac{1+x}{1-x} > 0 \Rightarrow -1 < x < 1$ $2z = \ln\left(\frac{1+x}{1-x}\right)$ $z = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$
$\operatorname{cosech}^{-1} x = \ln\left(\frac{1\pm\sqrt{1+x^2}}{x}\right)$, $x \ne 0$ Let $y = \operatorname{cosech}^{-1} x$, then $\operatorname{cosech} y = x$ $x = \frac{2}{e^y - e^{-y}} = \frac{2e^y}{e^{2y}-1}$ $x\, e^{2y} - 2\, e^y - x = 0$ $e^y = \frac{1\pm\sqrt{1+x^2}}{x} > 0$ When $x > 0$, $1 + x^2 > 1 \Rightarrow \frac{1-\sqrt{1+x^2}}{x} < 0$ (rejected) When $x < 0$, $\frac{1+\sqrt{1+x^2}}{x} < 0$ (rejected) $y = \operatorname{cosech}^{-1} x = \begin{cases} \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) & \text{if } x > 0 \\ \ln\left(\frac{1-\sqrt{1+x^2}}{x}\right) & \text{if } x < 0 \end{cases}$	$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, $0 < x \le 1$ Let $y = \operatorname{sech}^{-1} x$, then $\operatorname{sech} y = x$ $x = \frac{2}{e^y + e^{-y}} = \frac{2e^y}{e^{2y}+1}$ $x\, e^{2y} - 2\, e^y + x = 0$ $e^y = \frac{1\pm\sqrt{1-x^2}}{x} > 0 \Rightarrow -1 \le x \le 1, x \ne 0$ $\frac{1+\sqrt{1-x^2}}{x}\cdot\frac{1-\sqrt{1-x^2}}{x} = 1$ $\frac{1-\sqrt{1-x^2}}{x} = \left(\frac{1+\sqrt{1-x^2}}{x}\right)^{-1}$ When $x < 0$, $\frac{1\pm\sqrt{1-x^2}}{x} < 0$ (rejected) $y = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ or $-\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ It is convenient to adopt the positive sign and to write $y = \operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, $0 < x \le 1$

$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $\lvert x\rvert > 1$; let $y = \coth^{-1} x \Rightarrow x = \coth y = \frac{e^y + e^{-y}}{e^y - e^{-y}} = \frac{e^{2y}+1}{e^{2y}-1}$

$x\, e^{2y} - x = e^{2y} + 1 \Rightarrow e^{2y}(x - 1) = x + 1 \Rightarrow e^{2y} = \frac{x+1}{x-1} > 0$, $x < -1$ or $x > 1$

$y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $\lvert x\rvert > 1$

$\dfrac{\mathrm{d}\sinh x}{\mathrm{d}x}=\cosh x$ $\dfrac{\mathrm{d}\sinh x}{\mathrm{d}x}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{2}\left(e^{x}+e^{-x}\right)\right]$ $=\dfrac{1}{2}\left(e^{x}-e^{-x}\right)$ $=\cosh x$	$\dfrac{\mathrm{d}\cosh x}{\mathrm{d}x}=\sinh x$ $\dfrac{\mathrm{d}\cosh x}{\mathrm{d}x}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{2}\left(e^{x}-e^{-x}\right)\right]$ $=\dfrac{1}{2}\left(e^{x}+e^{-x}\right)$ $=\sinh x$
$\dfrac{\mathrm{d}\tanh x}{\mathrm{d}x}=\operatorname{sech}^2 x$ $\dfrac{\mathrm{d}\tanh x}{\mathrm{d}x}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right)$ $=\dfrac{\left(e^{x}+e^{-x}\right)^2-\left(e^{x}-e^{-x}\right)^2}{\left(e^{x}+e^{-x}\right)^2}$ $=\dfrac{4}{\left(e^{x}+e^{-x}\right)^2}$ $=\operatorname{sech}^2 x$	$\dfrac{\mathrm{d}\operatorname{cosech} x}{\mathrm{d}x}=-\operatorname{cosech} x\coth x$ $\dfrac{\mathrm{d}\operatorname{cosech} x}{\mathrm{d}x}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{2}{e^{x}-e^{-x}}\right)$ $=-\dfrac{2}{\left(e^{x}-e^{-x}\right)^2}\cdot\left(e^{x}+e^{-x}\right)$ $=-\dfrac{2}{e^{x}-e^{-x}}\cdot\dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$ $=-\operatorname{cosech} x\coth x$
$\dfrac{\mathrm{d}\operatorname{sech} x}{\mathrm{d}x}=-\operatorname{sech} x\tanh x$ $\dfrac{\mathrm{d}\operatorname{sech} x}{\mathrm{d}x}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{2}{e^{x}+e^{-x}}\right)$ $=-\dfrac{2}{\left(e^{x}+e^{-x}\right)^2}\cdot\left(e^{x}-e^{-x}\right)$ $=-\dfrac{2}{e^{x}+e^{-x}}\cdot\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$ $=-\operatorname{sech} x\tanh x$	$\dfrac{\mathrm{d}\coth x}{\mathrm{d}x}=-\operatorname{cosech} x$ $\dfrac{\mathrm{d}\coth x}{\mathrm{d}x}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\right)$ $=\dfrac{\left(e^{x}-e^{-x}\right)^2-\left(e^{x}+e^{-x}\right)^2}{\left(e^{x}-e^{-x}\right)^2}$ $=-\dfrac{4}{\left(e^{x}-e^{-x}\right)^2}$ $=-\operatorname{cosech}^2 x$
$\dfrac{\mathrm{d}\sinh^{-1} x}{\mathrm{d}x}=\dfrac{1}{\sqrt{1+x^2}}$ Let $y=\sinh^{-1} x$, then $\sinh y=x$ $\because\ \cosh^2 y-\sinh^2 y=1$ $\therefore\ \cosh y=\pm\sqrt{1+\sinh^2 y}=\pm\sqrt{1+x^2}$ $\cosh y\dfrac{\mathrm{d}y}{\mathrm{d}x}=1$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\cosh y}=\pm\dfrac{1}{\sqrt{1-x^2}}$ The slope of the graph of $y=\sinh x>0\ \forall\ x$ $\therefore\ \dfrac{\mathrm{d}\sinh^{-1} x}{\mathrm{d}x}=\dfrac{1}{\sqrt{1+x^2}}$	$\dfrac{\mathrm{d}\cosh^{-1} x}{\mathrm{d}x}=\dfrac{1}{\sqrt{x^2-1}}$ for all $x>1$ Let $y=\cosh^{-1} x$, then $\cosh y=x$ $\because\ \cosh^2 y-\sinh^2 y=1$ $\therefore\ \sinh y=\pm\sqrt{\cosh^2 y-1}=\pm\sqrt{x^2-1}$ $\sinh y\dfrac{\mathrm{d}y}{\mathrm{d}x}=1$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\sinh y}=\pm\dfrac{1}{\sqrt{x^2-1}}$ The slope of the graph of $y=\cosh x>0\ \forall\ x>1$ $\dfrac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x=\dfrac{1}{\sinh y}=\dfrac{1}{\sqrt{x^2-1}}$ for $x>1$

$\dfrac{\mathrm{d}\tanh^{-1}x}{\mathrm{d}x}=\dfrac{1}{1-x^2}$ for $|x| < 1$

Let $y = \tanh^{-1} x$, then $\tanh y = x$

$\operatorname{sech}^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\operatorname{sech}^2 y}$

$=\dfrac{1}{1-\tanh^2 y}$

$=\dfrac{1}{1-x^2}$

The slope of the graph of $y = \tanh x > 0\ \forall\ x$

$\therefore\ \dfrac{\mathrm{d}\tanh^{-1}x}{\mathrm{d}x}=\dfrac{1}{1-x^2}$ for $|x| < 1$

$\dfrac{\mathrm{d}\coth^{-1}x}{\mathrm{d}x}=\dfrac{1}{1-x^2}$, $|x| > 1$

Let $y = \coth^{-1} x$, then $\coth y = x$

$-\operatorname{cosech}^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-1}{\operatorname{cosech}^2 y}$

$=\dfrac{-1}{\coth^2 y-1}$

$=\dfrac{1}{1-x^2}$

The slope of the graph of $y = \coth x < 0\ \forall\ x$

$\therefore\ \dfrac{\mathrm{d}\coth^{-1}x}{\mathrm{d}x}=\dfrac{1}{1-x^2}$, $|x| > 1$

$\dfrac{\mathrm{d}\operatorname{sech}^{-1}x}{\mathrm{d}x}=-\dfrac{1}{x\sqrt{1-x^2}}$, $0 < x < 1$

Let $y = \operatorname{sech}^{-1} x$, then $\operatorname{sech} y = x$

$-\operatorname{sech} y \tanh y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-1}{\operatorname{sech} y\tanh y}$

$=\dfrac{-1}{\pm x\sqrt{1-\operatorname{sech}^2 y}}$

$=\dfrac{-1}{\pm x\sqrt{1-x^2}}$

The slope of the graph of $y = \cosh x > 0$, $0 < x < 1$

The slope of the graph of $y = \operatorname{sech} x < 0$, $0 < x < 1$

$\therefore\ \dfrac{\mathrm{d}\operatorname{cosech}^{-1}x}{\mathrm{d}x}=-\dfrac{1}{x\sqrt{1-x^2}}$, $0 < x < 1$

$\dfrac{\mathrm{d}\operatorname{cosech}^{-1}x}{\mathrm{d}x}=\dfrac{-1}{|x|\sqrt{1+x^2}}$ for $x \neq 0$

Let $y = \operatorname{cosech}^{-1} x$, then $\operatorname{cosech} y = x$

$-\operatorname{cosech} y \coth y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-1}{\operatorname{cosech} y\coth y}$

$=\dfrac{-1}{\pm x\sqrt{1+\operatorname{cosech}^2 y}}$

$=\dfrac{-1}{\pm x\sqrt{1+x^2}}$

The slope of the graph of $y = \sinh x > 0\ \forall\ x$

The slope of the graph of $y = \operatorname{cosech} x < 0$, $x \neq 0$

$\therefore\ \dfrac{\mathrm{d}\operatorname{cosech}^{-1}x}{\mathrm{d}x}=\dfrac{-1}{|x|\sqrt{1+x^2}}$, $x \neq 0$

$\displaystyle\int\frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x=\ln\left|x+\sqrt{x^2+1}\right|+\mathrm{c} = \sinh^{-1} x + \mathrm{c}$

$\because\ \dfrac{\mathrm{d}\sinh^{-1}x}{\mathrm{d}x}=\dfrac{1}{\sqrt{1+x^2}}$

$\therefore\ \displaystyle\int\frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x=\sinh^{-1} x + \mathrm{c}$

$=\ln\left|x+\sqrt{x^2+1}\right|+\mathrm{c}$

$\displaystyle\int\frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x=\ln\left|x+\sqrt{x^2-1}\right|+\mathrm{c} = \cosh^{-1} x + \mathrm{c}$

$\because\ \dfrac{\mathrm{d}\cosh^{-1}x}{\mathrm{d}x}=\dfrac{1}{\sqrt{x^2-1}}$, for all $x > 1$

$\therefore\ \displaystyle\int\frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x=\cosh^{-1} x + \mathrm{c}$

$=\ln\left(x+\sqrt{x^2-1}\right)+\mathrm{c}$

$\displaystyle\int\frac{1}{1-x^2}\,\mathrm{d}x=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+\mathrm{c} = \tanh^{-1} x + \mathrm{c}$

$\because\ \dfrac{\mathrm{d}\tanh^{-1}x}{\mathrm{d}x}=\dfrac{1}{1-x^2}$, for $|x| < 1$

$\therefore\ \displaystyle\int\frac{1}{1-x^2}\,\mathrm{d}x=\tanh^{-1} x + \mathrm{c} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)+\mathrm{c}$

$\displaystyle\int\frac{1}{x^2-1}\,\mathrm{d}x=-\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+\mathrm{c} = -\coth^{-1} x + \mathrm{c}$

$\because\ \dfrac{\mathrm{d}\coth^{-1}x}{\mathrm{d}x}=\dfrac{1}{1-x^2}$, $|x| > 1$

$\displaystyle\int\frac{1}{x^2-1}\,\mathrm{d}x=-\coth^{-1} x + \mathrm{c} = -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+\mathrm{c}$

$\int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x = -\ln\left\lvert\frac{1+\sqrt{1-x^2}}{x}\right\rvert + \mathrm{c} = -\mathrm{sech}^{-1}\,x + \mathrm{c}$ $\because\ \frac{\mathrm{d}\,\mathrm{sech}^{-1}x}{\mathrm{d}x} = -\frac{1}{x\sqrt{1-x^2}}$, $0 < x < 1$ $\int \frac{1}{x\sqrt{1-x^2}}\,\mathrm{d}x = -\mathrm{sech}^{-1}\,x + \mathrm{c}$ $= -\ln\left\lvert\frac{1+\sqrt{1-x^2}}{x}\right\rvert + \mathrm{c}$	$\int \frac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x = -\ln\left\lvert\frac{1+\sqrt{1+x^2}}{x}\right\rvert + \mathrm{c} = -\mathrm{cosech}^{-1}\,x + \mathrm{c}$ $\because\ \frac{\mathrm{d}\,\mathrm{cosech}^{-1}x}{\mathrm{d}x} = \frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$, for $x \neq 0$ $\int \frac{1}{x\sqrt{1+x^2}}\,\mathrm{d}x = -\mathrm{cosech}^{-1}\,x + \mathrm{c}$ $= -\ln\left\lvert\frac{1+\sqrt{1+x^2}}{x}\right\rvert + \mathrm{c}$