2 straight lines Invariants

Created by Mr. Francis Hung on 20081112
Last updated: August 30, 2021

Apply rotation θ to the lines $ax^2 + 2hxy + by^2 = 0$.

$x = x_1 \cos\theta - y_1 \sin\theta$

$y = x_1 \sin\theta + y_1 \cos\theta$

Let the new rotated equation be $a_1x_1^2 + 2h_1x_1y_1 + b_1y_1^2 = 0$

Prove that $a_1 + b_1 = a + b$ and $a_1b_1 - h_1^2 = ab - h^2$.

Apply rotation θ to $ax^2 + 2hxy + by^2 = 0$

$x = x_1 \cos\theta - y_1 \sin\theta$

$y = x_1 \sin\theta + y_1 \cos\theta$

$a(x_1 \cos\theta - y_1 \sin\theta)^2 + 2h(x_1 \cos\theta - y_1 \sin\theta)(x_1 \sin\theta + y_1 \cos\theta) + b(x_1 \sin\theta + y_1 \cos\theta)^2 = 0$

$a_1 = a \cos^2\theta + 2h \sin\theta \cos\theta + b \sin^2\theta$

$= a\dfrac{1+\cos 2\theta}{2} + h\sin 2\theta + b\dfrac{1-\cos 2\theta}{2}$

$= \dfrac{a+b}{2} + h\sin 2\theta + \dfrac{a-b}{2}\cos 2\theta$

$h_1 = -a\cos\theta\sin\theta + h(\cos^2\theta - \sin^2\theta) + b\cos\theta\sin\theta$

$= \dfrac{b-a}{2}\sin 2\theta + h\cos 2\theta$

$b_1 = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$

$= a\dfrac{1-\cos 2\theta}{2} - h\sin 2\theta + b\dfrac{1+\cos 2\theta}{2}$

$= \dfrac{a+b}{2} - h\sin 2\theta - \dfrac{a-b}{2}\cos 2\theta$

$a_1 + b_1 = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta + a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta = a + b$

$a_1b_1 - h_1^2 = \left[\dfrac{a+b}{2} + \left(h\sin 2\theta + \dfrac{a-b}{2}\cos 2\theta\right)\right]\left[\dfrac{a+b}{2} - \left(h\sin 2\theta + \dfrac{a-b}{2}\cos 2\theta\right)\right] - [\dfrac{b-a}{2}\sin 2\theta + h\cos 2\theta]^2$

$= \left(\dfrac{a+b}{2}\right)^2 - \left(h\sin 2\theta + \dfrac{a-b}{2}\cos 2\theta\right)^2 - \left(\dfrac{b-a}{2}\right)^2\sin^2 2\theta - (b-a)h\sin 2\theta\cos 2\theta - h^2\cos^2 2\theta$

$= \left(\dfrac{a+b}{2}\right)^2 - h^2\sin^2 2\theta - (a-b)h\sin 2\theta\cos 2\theta - \left(\dfrac{a-b}{2}\right)^2\cos^2 2\theta - \left(\dfrac{b-a}{2}\right)^2\sin^2 2\theta + (a-b)h\sin 2\theta\cos 2\theta - h^2\cos^2 2\theta$

$= \left(\dfrac{a+b}{2}\right)^2 - \left(\dfrac{a-b}{2}\right)^2 - h^2 = ab - h^2$