Two Straight Lines

Reference: Advanced Level Pure Mathematics by S.L. Green p.27-p36

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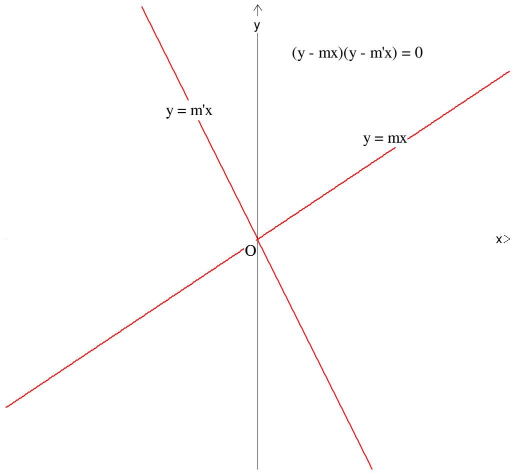
1. Consider $(y - mx)(y - m'x) = 0 \cdots\cdots (1)$

(1) represents a pair of straight lines passes through the origin with slope = m, m'.

Expand (1) in <u>general</u> form: $ax^2 + 2hxy + by^2 = 0 \cdots\cdots (2)$

If $b \neq 0$, then $\left(\frac{y}{x}\right)^2 + 2\frac{h}{b}\left(\frac{y}{x}\right) + \frac{a}{b} = 0$, which is a quadratic equation in $\frac{y}{x}$.

It gives 2 roots m, m'. The roots are real iff $\left(\frac{h}{b}\right)^2 - \frac{a}{b} \geq 0 \Leftrightarrow h^2 - ab \geq 0 \because b^2 > 0$



It may be rewritten as $\left(\frac{y}{x} - m\right)\left(\frac{y}{x} - m'\right) = 0$

$\Rightarrow (y - mx)(y - m'x) = 0$

Sum of roots = $m + m' = -\frac{2h}{b}$, product of roots = $m\,m' = \frac{a}{b}$

(Note: if $b = 0$, (2) becomes $x(ax + 2hy) = 0$)

2. **<u>The angle between the lines</u>**

Suppose θ = angle between the lines.

$$\tan\theta = \left|\frac{m - m'}{1 + mm'}\right|$$

$$\tan^2\theta = \frac{(m + m')^2 - 4mm'}{(1 + mm')^2} = \frac{\left(-\frac{2h}{b}\right)^2 - 4\frac{a}{b}}{\left(1 + \frac{a}{b}\right)^2}$$

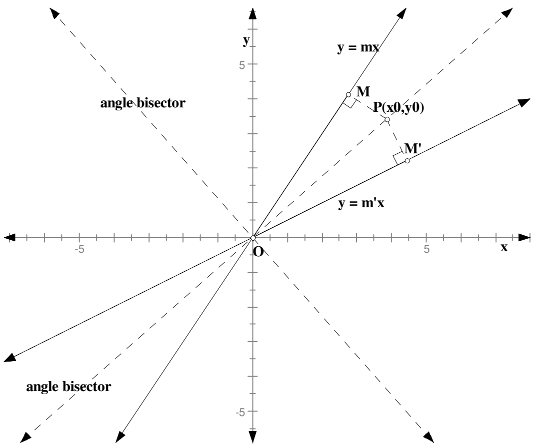
$$\tan^2\theta = \frac{4(h^2 - ab)}{(a + b)^2} \Rightarrow \tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{(a + b)}$$

If θ is acute, choose the sign $\pm$ so that $\tan\theta > 0$

If $\theta = 0$, the 2 lines coincide if $h^2 - ab = 0$

If $\theta = \frac{\pi}{2}$, the lines are perpendicular if $a + b = 0$

3. **<u>The angle bisectors</u>**



Let P(x_0, y_0) be any point on the angle bisector.

Let M, M' be the feet of perpendicular onto $y = mx$, $y = m'x$ respectively, then $\Delta OPM \cong \Delta OPM'$ (AAS)

$PM = PM'$

$$\frac{mx_0 - y_0}{\sqrt{1 + m^2}} = \pm\frac{m'x_0 - y_0}{\sqrt{1 + m'^2}}$$

The two angle bisectors are $\frac{mx-y}{\sqrt{1+m^2}}-\frac{m'x-y}{\sqrt{1+m'^2}}=0$ and $\frac{mx-y}{\sqrt{1+m^2}}+\frac{m'x-y}{\sqrt{1+m'^2}}=0$

Multiply together to give the pair of straight lines.

$$\frac{(mx-y)^2}{1+m^2}-\frac{(m'x-y)^2}{1+m'^2}=0$$

$(1 + m'^2)(m^2x^2 - 2mxy + y^2) - (1 + m^2)(m'^2x^2 - 2m'xy + y^2) = 0$

$(m^2 - m'^2)x^2 + 2(m' - m)(1 - mm')xy + (m'^2 - m^2)y^2 = 0$

$-(m' + m)x^2 + 2(1 - mm')xy + (m' + m)y^2 = 0$

$$-\frac{-2h}{b}x^2+2\left(1-\frac{a}{b}\right)xy-\frac{2h}{b}y^2=0$$

$hx^2 - (a - b)xy - hy^2 = 0$

Example 1 Find the equation of 2 straight lines through the origin and are perpendicular to $ax^2 + 2hxy + by^2 = 0$

Let $y = m_3x$, $y = m_4x$ be the two straight lines which are $\perp$ to $y = mx$ and $y = m'x$ respectively.

Then $m_3=-\frac{1}{m}$, $m_4=-\frac{1}{m'}$

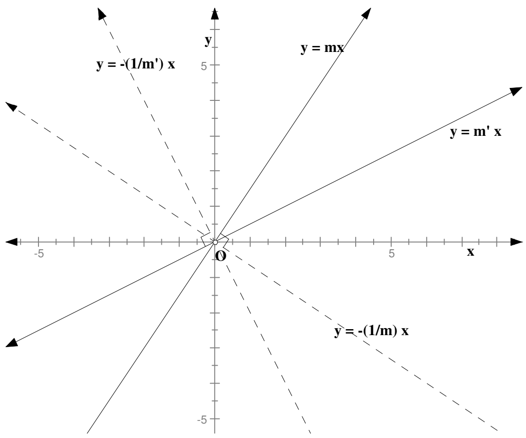
$\Rightarrow y=-\frac{1}{m}x,\ y=-\frac{1}{m'}x$

$(x + my)(x + m'y) = 0$

$x^2 + (m + m')xy + mm'y^2 = 0$

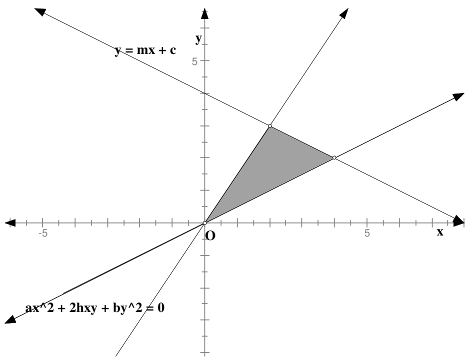
$$x^2-\frac{2h}{b}xy+\frac{a}{b}y^2=0$$

$bx^2 - 2hxy + ay^2 = 0$



Exercise 1 Find the area of the triangle formed by $ax^2 + 2hxy + by^2 = 0$ and $y = mx + c$.

Answer: $\frac{c^2\sqrt{h^2-ab}}{\left|a+2hm+bm^2\right|}$



4. Change of origin

Given the equation $ax^2 + 2hxy + by^2 = 0$

Consider the translation $x = \text{X} + p$, $y = \text{Y} + q$.

The pair of straight lines will be transformed into

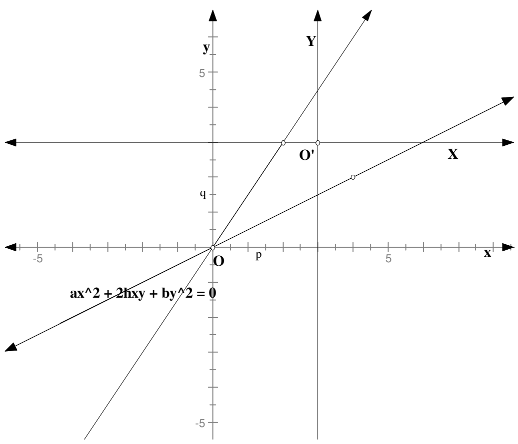
$a(\text{X} + p)^2 + 2h(\text{X} + p)(\text{Y} + q) + b(\text{Y} + q)^2 = 0$

After expansion and grouping like terms,

$a\text{X}^2 + 2h\text{XY} + b\text{Y}^2 + 2g\text{X} + 2f\text{Y} + c = 0$

where $g = ap + hq$, $f = hp + bq$, $c = ap^2 + 2hpq + bq^2$

It is the general equation of second degree, represent a pair of straight lines.



5. <u>Condition for a general equation of second degree to represent a pair of straight lines</u>

Consider $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \cdots\cdots (*)$

Suppose it represent a pair of straight lines which intersect at (p, q)

Perform the translation $x = X + p$, $y = Y + q$

$a(X + p)^2 + 2h(X + p)(Y + q) + b(Y + q)^2 + 2g(X + p) + 2f(Y + q) + c = 0$

Since it represents a pair of straight lines passes through the new origin, it must be reduced to $aX^2 + 2hXY + bY^2 = 0$

coefficient of X = 0 $\Rightarrow ap + hq + g = 0 \cdots\cdots (1)$

coefficient of Y = 0 $\Rightarrow hp + bq + f = 0 \cdots\cdots (2)$

constant = 0 $\Rightarrow ap^2 + 2hpq + bq^2 + 2gp + 2fq + c = 0 \cdots\cdots (3)$

(3): $p(ap + hq + g) + q(hp + bq + f) + gp + fq + c = 0$

$\Rightarrow gp + fq + c = 0 \cdots\cdots (4)$

Solving (1) and (2): $\dfrac{p}{hf - bg} = \dfrac{q}{gh - af} = \dfrac{1}{ab - h^2} \quad \cdots\cdots (5)$

Sub. p, q from (5) into (4): $g(hf - bg) + f(gh - af) + c(ab - h^2) = 0$

$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

It may be rewritten as $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$, which is called the <u>discriminant</u>.

Equation (5) determines (p, q) provided that $ab - h^2 \neq 0$.

If equation (*) represents a pair of parallel straight lines, then it is equivalent to:

$(mx - y + c_1)(mx - y + c_2) = 0$

$(mx - y)^2 + (c_1 + c_2)(mx - y) + c_1c_2 = 0$

Expand it and compare it with (*) gives:

$\dfrac{a}{b} = m^2$, $\dfrac{h}{b} = -m$, $\dfrac{2g}{b} = m(c_1 + c_2)$, $\dfrac{2f}{b} = -(c_1 + c_2)$, $\dfrac{c}{b} = c_1c_2$; provided that $b \neq 0$

$ab - h^2 = b^2m^2 - b^2m^2 = 0$

$abc + 2fgh - af^2 - bg^2 - ch^2$

$= b^3m^2c_1c_2 + b(c_1 + c_2)\dfrac{bm(c_1 + c_2)}{2}bm - \dfrac{b^3m^2(c_1 + c_2)^2}{4} - \dfrac{b^3m^2(c_1 + c_2)^2}{4} - b^3c_1c_2m^2 = 0$

Therefore $\Delta = 0$

Conclusion: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

<table>
<tr><td colspan="2">$\Delta = 0$</td><td>$\Delta \neq 0$</td></tr>
<tr><td>$ab - h^2 \neq 0$</td><td>$ab - h^2 = 0$</td><td rowspan="2">It is <u>NOT</u> a pair of straight lines.</td></tr>
<tr><td>It represents a pair of straight lines intersect at (p, q).
$p = \dfrac{hf - bg}{ab - h^2}$, $q = \dfrac{gh - af}{ab - h^2}$</td><td>It represents a pair of parallel lines.
Slope $= \pm\sqrt{\dfrac{a}{b}}$.
Sign is taken as the opposite of h.</td></tr>
</table>

Example 2 Consider $x^2 + 2xy + y^2 - 2x - 2y - 15 = 0$

Here $a = 1$, $h = 1$, $b = 1$, $g = -1$, $f = -1$, $c = -15$

It is easy to show that $ab - h^2 = 0$ and $\Delta = 0$

$\therefore$ It represents a pair of parallel lines.

Completing the squares $(x + y)^2 - 2(x + y) - 15 = 0$

$(x + y - 5)(x + y + 3) = 0$

Exercise 2 Consider $x^2 - 4xy + y^2 - 10x + 8y + 13 = 0$

(a) Prove that it represent a pair of straight lines;

(b) Find the point of intersection.

(c) Find the angle between them.

Answers (b) (1, –2), (c) $\theta = \dfrac{\pi}{3}$

Exercise 3 (a) Find the value of c for which $11x^2 + 20xy - 4y^2 - 84x - 24y + c = 0$ represents a pair of straight lines;

(b) find their separate equations;

(c) find the equation of bisectors of the angle between these lines.

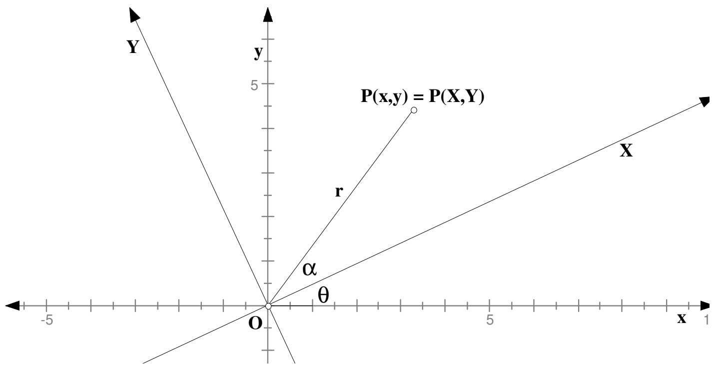
Answers (a) $c = 108$; (b) $11x - 2y - 18 = 0$, $x + 2y - 6 = 0$, (c) $(11x-2y-18)^2 - 25(x+2y-6)^2 = 0$

6. <u>Rotation of axes by θ (in anti-clockwise direction)</u>

Let $OP = r$, $\angle XOx = \theta$, $\angle POX = \alpha$, then $\angle POX = \theta + \alpha$.

$$\begin{cases} X = r\cos\alpha \\ Y = r\sin\alpha \end{cases}, \quad \begin{cases} x = r\cos(\theta+\alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha) \\ y = r\sin(\theta+\alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \end{cases}$$

$$\Rightarrow \begin{cases} x = X\cos\theta - Y\sin\theta \\ y = X\sin\theta + Y\cos\theta \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$



Example 3 Consider $15x^2 - 16xy - 48y^2 = 0$

(a) Rotate the axes by θ;

(b) find θ so that the product term is absent from the equation.

(a) $15(X\cos\theta - Y\sin\theta)^2 - 16(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) - 48(X\sin\theta + Y\cos\theta)^2 = 0$

(b) coefficient of XY is $-30\cos\theta\sin\theta - 16(\cos^2\theta - \sin^2\theta) - 96(\sin\theta\cos\theta) = 0$

$8\sin^2\theta - 63\cos\theta\sin\theta - 8\cos^2\theta = 0$

$(8\sin\theta + \cos\theta)(\sin\theta - 8\cos\theta) = 0$

$\tan\theta = 8$ or $-\dfrac{1}{8} \Rightarrow \theta = 82°54'$ or $172°54'$.

7. <u>**Invariants**</u> Apply the rotation by θ to $ax^2 + 2hxy + by^2 = 0$

$a(\text{X}\cos\theta - \text{Y}\sin\theta)^2 + 2h(\text{X}\cos\theta - \text{Y}\sin\theta)(\text{X}\sin\theta + \text{Y}\cos\theta) + b(\text{X}\sin\theta + \text{Y}\cos\theta)^2 = 0$

It is in the form $a'\text{X}^2 + 2h'\text{XY} + b'\text{Y}^2 = 0$

where $a' = a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta$,

$h' = -a\cos\theta\sin\theta + h(\cos^2\theta - \sin^2\theta) + b\cos\theta\sin\theta$,

$b' = a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta$.

It can be easily shown that $a' + b' = a + b$ ⋯⋯ (1)

$$a' = a\frac{1+\cos 2\theta}{2} + h\sin 2\theta + b\frac{1-\cos 2\theta}{2} = \frac{a+b}{2} + \left(\frac{a-b}{2}\cos 2\theta + h\sin 2\theta\right)$$

$$b' = a\frac{1-\cos 2\theta}{2} - h\sin 2\theta + b\frac{1+\cos 2\theta}{2} = \frac{a+b}{2} - \left(\frac{a-b}{2}\cos 2\theta + h\sin 2\theta\right)$$

$$h' = -\frac{1}{2}(a-b)\sin 2\theta + h\cos 2\theta$$

$$a'b' - h'^2 = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\cos 2\theta + h\sin 2\theta\right)^2 - \left[-\frac{1}{2}(a-b)\sin 2\theta + h\cos 2\theta\right]^2$$

$$= \left(\frac{a+b}{2}\right)^2 - \left[\left(\frac{a-b}{2}\right)^2 + h^2\right]$$

$a'b' - h'^2 = ab - h^2$ ⋯⋯ (2)

(1) and (2) are the invariants of the rotation.

Example 4 Find the equation of the pair of straight lines joining the origin to the intersection of $\ell x + my + n = 0$ with the curve $ax^2 + by^2 = 1$.

Let (x, y) be the intersection point, then

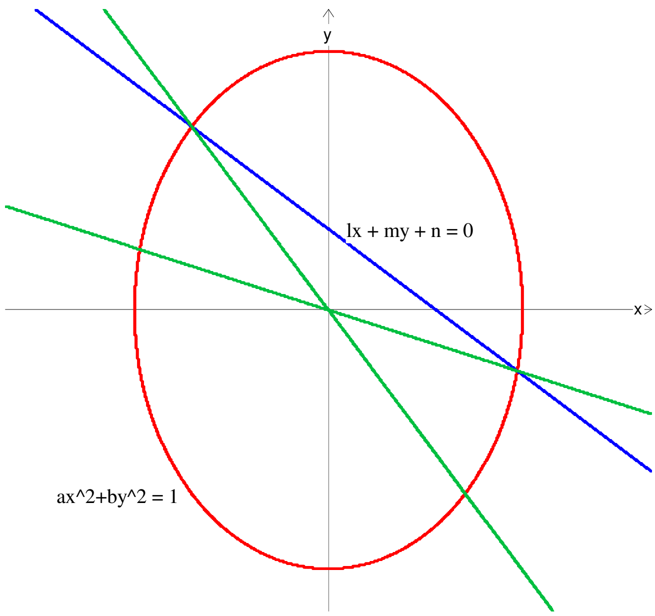
$$\begin{cases} ax^2 + by^2 = 1 \cdots\cdots (1) \\ -\dfrac{\ell x + my}{n} = 1 \cdots\cdots (2) \end{cases}$$

$(1) = (2)^2 \quad ax^2 + by^2 = \left(\dfrac{\ell x + my}{n}\right)^2$

It is a second degree equation in the form $Ax^2 + 2Hxy + By^2 = 0$

Also it passes through the intersection points.

∴ It represents pair of straight lines through the origin and the intersection points.



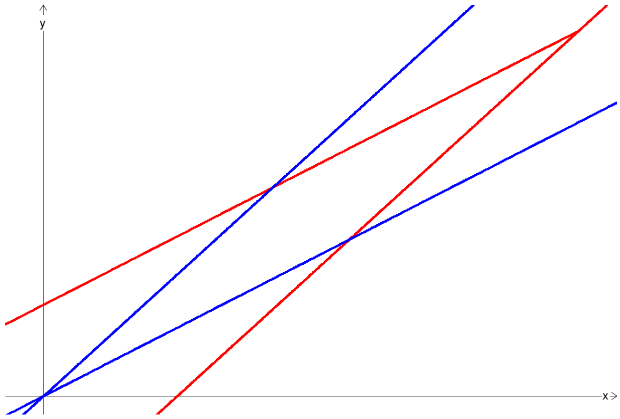
Exercise 4

(a) Show that S: $12x^2 - 25xy + 12y^2 + 7x + 7y - 49 = 0$ represents a pair of straight lines;

(b) Another pair of straight lines are drawn through the origin and parallel to those of S. A parallelogram is formed.

(i) Show that the parallelogram is a rhombus;

(ii) determine its area.



Ans (b) (i) Vertices (0,0), (4,3), (7,7), (3,4).

(ii) area = 7 sq. units.

Example 5 Given $22x^2 - 15xy + 2y^2 = 0$.

Find the equation of the pair of straight lines referred to its angle bisectors as axes.

Apply the rotation by θ.

$22(X\cos\theta - Y\sin\theta)^2 - 15(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + 2(X\sin\theta + Y\cos\theta)^2 = 0 \cdots(*)$

Now the equation of lines referred to the angle bisectors is of the form $(Y - mX)(Y + mX) = 0$

$\Rightarrow Y^2 - m^2X^2 = 0$, coefficient of $XY = 0$

$\Rightarrow -44\cos\theta\sin\theta - 15(\cos^2\theta - \sin^2\theta) + 4\cos^2\theta\sin\theta = 0$

$3\tan^2\theta - 8\tan\theta - 3 = 0$

$(3\tan\theta + 1)(\tan\theta - 3) = 0$

$\tan\theta = -\frac{1}{3}$ or 3.

when $\tan\theta = -\frac{1}{3}$, (*) becomes $22(X - Y\tan\theta)^2 - 15(X - Y\tan\theta)(X\tan\theta + Y) + 2(X\tan\theta + Y)^2 = 0$

$22(X + \frac{1}{3}Y)^2 - 15(X + \frac{1}{3}Y)(-\frac{1}{3}X + Y) + 2(-\frac{1}{3}X + Y)^2 = 0$

$22(9X^2 + 6XY + Y^2) - 15(8XY - 3X^2 + 3Y^2) + 2(X^2 - 6XY + 9Y^2) = 0$

$245X^2 - 5Y^2 = 0 \Rightarrow 49X^2 - Y^2 = 0$

when $\tan\theta = 3$, $22(X - 3Y)^2 - 15(X - 3Y)(3X + Y) + 2(3X + Y)^2 = 0$

$\Rightarrow X^2 - 49Y^2 = 0$