1. Standard conics (centre $= (0,0)$, axes $= x$-y axis)

1.1 2 st. lines. $ax^2+2hxy+by^2=0$

slopes of the 2 lines m_1, m_2 are real iff $h^2-ab \geq 0$

1.2 circle $(\frac{x}{r})^2+(\frac{y}{r})^2=1$

1.3 ellipse $(\frac{x}{a})^2+(\frac{y}{b})^2=1$

1.4 hyperbola $(\frac{x}{a})^2-(\frac{y}{b})^2=\pm 1$

1.5 parabola $y^2=4ax$ or $x^2=4ay$

Apart from the equation of parabola (1.5), general equation of standard conics (1.2), (1.3), (1.4) is:

$$Ax^2+By^2=1$$

clearly A and B cannot be both negative, otherwise no locus.

if $A>0$ $B>0$ and $A=B$ then it is a circle

if $A>0$ $B>0$ and $A \neq B$ then it is an ellipse

if $AB<0$ then it is an hyperbola

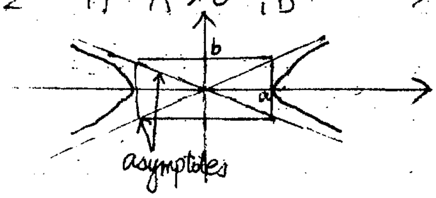
1.6 In the case $AB<0$, the asymptotes is: $Ax^2+By^2=0$

pf: 1° $\because AB<0 \Rightarrow A>0, B<0$ or $A<0, B>0$

from 1.1 the condition $h^2-ab = 0^2-AB>0$

$\therefore Ax^2+By^2=0$ is a pair of st. lines through origin

2° if $A>0, B<0 \Rightarrow (\frac{x}{a})^2-(\frac{y}{b})^2=1$, $a=\frac{1}{\sqrt{A}}$, $b=\frac{1}{\sqrt{-B}}$



slopes of 2 asymptotes are $\frac{b}{a}, -\frac{b}{a}$.

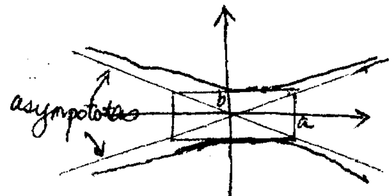
$\therefore$ eqts: $y=\frac{b}{a}x$ $y=-\frac{b}{a}x$.

$\Rightarrow (ay-bx)(ay+bx)=0$

$\Rightarrow a^2y^2-b^2x^2=0$

$\Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \Rightarrow Ax^2+By^2=0$

3° if $A<0, B>0 \Rightarrow (\frac{y}{b})^2-(\frac{x}{a})^2=1$ $a=\frac{1}{\sqrt{-A}}$, $b=\frac{1}{\sqrt{B}}$



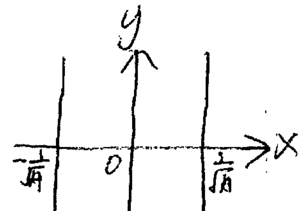
similar arguements lead to

$(\frac{y}{b})^2-(\frac{x}{a})^2=0 \Rightarrow Ax^2+By^2=0$

1.7 It is worthwhile to note that the eqt of asymptotes differ from the hyperbola by a $\boxed{\text{constant}}$.

1.8 if $A>0,\ B=0$ $\quad Ax^2+By^2=1 \Rightarrow Ax^2=1$

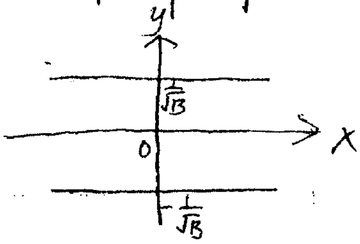
$\Rightarrow x=\pm\frac{1}{\sqrt{A}}$



$\therefore$ it is 2 vertical lines

if $B>0$, $A=0$ $\quad Ax^2+By^2=1 \Rightarrow By^2=1$

$\Rightarrow y=\pm\frac{1}{\sqrt{B}}$



$\therefore$ it is 2 horizontal lines

if $A=B=0$ no locus.

Conclusion : $Ax^2+By^2=1$

conditions	$AB>0$			$AB=0$					$AB<0$
	$A>0, B>0$		$A<0\ B<0$	$A=0$		$B=0$			asymptotes:
	$A=B$	$A\neq B$		$B>0$	$B<0$	$A=0$ $B=0$	$A>0$	$A<0$	$Ax^2\pm By^2=0$
Locus name	circle	ellipse	no locus	2 horizontal lines	no locus	no locus	2 vertical lines	no locus	hyperbola
examples:	$2x^2+2y^2=1$	$2x^2+3y^2=1$	$-x^2-5y^2=1$	$2y^2=1$	$-3y^2=1$	$0=1$	$2x^2=1$	$-5x^2=1$	$2x^2-3y^2=1$

1.9 Intersection of ~~general~~ central conics to straight line.

$$\begin{cases} Ax^2+By^2=1 \\ y=mx+c \end{cases}$$

In general, there are at most 2 solutions.

If the solutions are distinct, then the line $y=mx+c$ is called a 'chord'

The line $y=mx$ is called the diameter

1.10 let (x_0,y_0) be a point on $Ax^2+By^2=1$

The equation of tangent is: $Ax_0x+By_0y=1$ —①

(can be proved by differentiation).

Equation of normal at (x_0,y_0) is: $\frac{x-x_0}{Ax_0}=\frac{y-y_0}{By_0}$

1.11 Condition for tangency: $lx+my+n=0$ —②

① and ② are essentially the same $\therefore \frac{Ax_0}{l}=\frac{By_0}{m}=\frac{-1}{n}$

$\Rightarrow x_0=-\frac{l}{An}$, $y_0=-\frac{m}{Bn}$ $\quad Ax_0^2+By_0^2=1$

$\Rightarrow A\left(-\frac{l}{An}\right)^2+B\left(-\frac{m}{Bn}\right)^2=1 \Rightarrow \frac{l^2}{A}+\frac{m^2}{B}=n^2$

Condition for $y=mx+c$ to be a tangent ·

$y=mx+c$ and $AX_0X + BY_0Y = 1$ are equivalent

$$\Rightarrow \frac{AX_0}{m} = \frac{BY_0}{-1} = \frac{-1}{c}$$

$$\Rightarrow X_0 = -\frac{m}{AC} \qquad Y_0 = \frac{1}{BC}$$

$$\Rightarrow \because AX_0^2+BY_0^2=1 \quad \therefore \quad A\left(-\frac{m}{AC}\right)^2 + B\left(\frac{1}{BC}\right)^2 = 1$$

$$\Rightarrow \frac{m^2}{A} + \frac{1}{B} = c^2$$

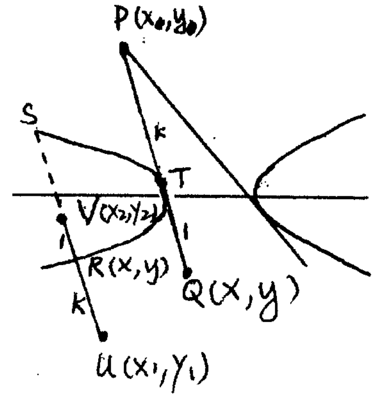
$$\Rightarrow c = \pm\sqrt{\frac{m^2}{A} + \frac{1}{B}}$$

$$\Rightarrow y = mx \pm \sqrt{\frac{m^2}{A} + \frac{1}{B}}$$

eg find two equations of straight lines with slope $=2$ and touch $2x^2+3y^2=1$

Sol. $y = 2x \pm \sqrt{\frac{4}{2}+\frac{1}{3}}$

$y = 2x \pm \sqrt{\frac{7}{3}}$



1.12 Tangents from a point

Let $U(x_1,y_1)$, $V(x_2,y_2)$ be 2 points.

If U, V cut the curve $Ax^2+By^2=1$ at R, S

suppose $UR : RV = k : 1$

then $R = \left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}\right)$ —— ①

$\because$ R lies on the curve

$\therefore A\left(\frac{x_1+kx_2}{1+k}\right)^2 + B\left(\frac{y_1+ky_2}{1+k}\right)^2 = 1$

which can be simplify to —②

$$(Ax_2^2+By_2^2-1)k^2 + 2(Ax_1x_2+By_1y_2-1)k + (Ax_1^2+By_1^2-1)=0$$

This is a quadratic equation in k, in general has 2 solutions

$\therefore$ we can find 2 values of k (k_1, k_2)

after we have subsituted k_1 (k_2 respectively) to ①

we find 2 points R and S on the curve.

If $R=S$ then UV is a tangent and ② has equal roots

In this case $\Delta=0$

$$\therefore (Ax_1^2+By_1^2-1)(Ax_2^2+By_2^2-1) = (Ax_1x_2+By_1y_2-1)^2$$

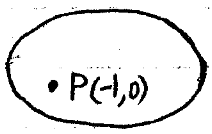
Suppose $U(x_1, y_1) = P(x_0, y_0)$, $V(x_2, y_2) = Q(x, y)$ where $PT : TQ = k : 1$, T lies on the curve.

Then the equation of pair of straight lines is:

$$(Ax_0^2 + By_0^2 - 1)(Ax^2 + By^2 - 1) = (Ax_0x + By_0y - 1)^2$$

Note that the necessary condition is : $\boxed{P(x_0, y_0) \text{ lies outside}}$

Please try the above equation to



$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{1}\right)^2 = 1$$

.13 <u>Equation of asymptotes</u>

Suppose $AB < 0$ then $Ax^2 + By^2 = 1$ is a hyperbola

if $P(x_0, y_0) = O(0,0)$ the origin.

then the equation of pair of tangents at $O(0,0)$ is

$$(A0^2 + B0^2 - 1)(Ax^2 + By^2 - 1) = (A0x + B0y - 1)^2$$

$$Ax^2 + By^2 - 1 = -1$$

$$\boxed{Ax^2 + By^2 = 0}$$

using ② in section 1.12

$$k = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$k = \frac{-b \pm \sqrt{0}}{2a}$$

$$k = \frac{-(Ax_1x_2 + By_1y_2 - 1)}{Ax_2^2 + By_2^2 - 1}$$

$$= -\frac{-1}{Ax^2 + By^2 - 1} \qquad \begin{pmatrix} U(x_1, y_1) = P(0,0) \\ V(x_2, y_2) = Q(x,y) \end{pmatrix}$$

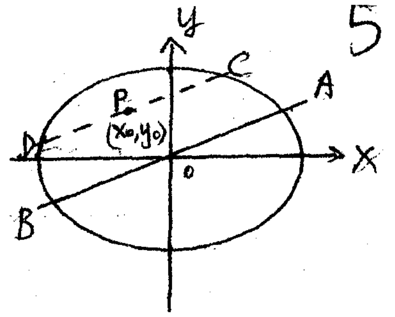
$$= -1 \qquad (\because Ax^2 + By^2 = 0)$$

$\Rightarrow$ which is possible only if T lies at infinity

$\therefore$ $Ax^2 + By^2 = 0$ is a pair of asymptotes.

Please refer to section 1.6

1.14 Conjugate diameters



AB is a line passes through origin.
It is called a diameter (section 1.6)

Let the slope of AB be m_{AB}
The line CD parallel to AB (slope $= m_{AB}$) called a chord
Suppose $P(x_0, y_0)$ is the mid point of CD
then the equation of CD is $\begin{cases} x = x_0 + t \\ y = y_0 + m_{AB} t \end{cases}$, t is a parameter

$\because$ C, D lie on the line and on the curve $Ax^2 + By^2 = 1$
$\therefore$ $A(x_0+t)^2 + B(y_0 + m_{AB} t)^2 = 1$
$\Rightarrow (A + B m_{AB}^2) t^2 + 2(Ax_0 + B m_{AB} y_0) t + (Ax_0^2 + By_0^2 - 1) = 0$
This is a quadratic equation in t, and has 2 values t_1, t_2
$\because$ P is the mid point
$\therefore$ t_1 and t_2 are numerically the same and opposite in sign
$\therefore$ $t_1 = -t_2$
$\Rightarrow$ $t_1 + t_2 = 0$
$\Rightarrow$ $Ax_0 + B m_{AB} y_0 = 0$
$\therefore$ The equation of locus of mid points of chord parallel to AB is : $Ax + B m_{AB} y = 0$
It is called the conjugate diameter of AB

A conjugate diameter has a slope $-\frac{A}{B m_{AB}}$.

Product of slopes of conjugate diameters $= -\frac{A}{B m_{AB}} \times m_{AB}$

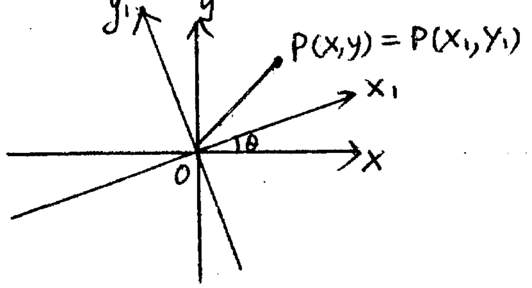
$$= -\frac{A}{B}$$

eg. Given $-2x^2 + 3y^2 = 1$ a hyperbola
A diameter is: $x + 4y = 0$ with slope $= -\frac{1}{4}$
Then the conjugate diameter is: $y = -\frac{-2}{3 \times (-\frac{1}{4})} x$

ie $8x + 3y = 0$ //

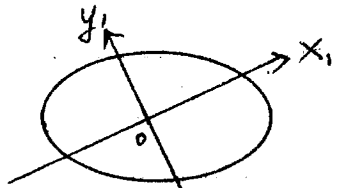
2. Central Conics

Suppose $ax^2+by^2=1$ is an ellipse or a hyperbola



If we rotate the co-ordinate axes in anticlockwise direction θ we shall find that (with a little manipulation),

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$



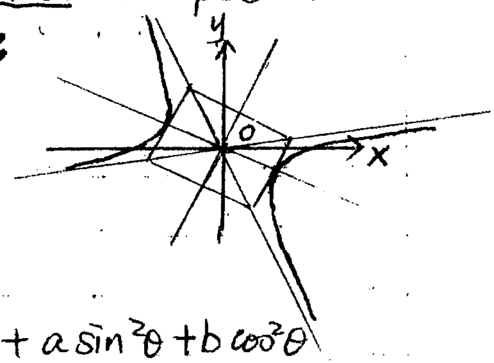
After rotation, the equation becomes

$$a(x_1\cos\theta - y_1\sin\theta)^2 + b(x_1\sin\theta + y_1\cos\theta)^2 = 1$$

$$(a\cos^2\theta + b\sin^2\theta)\,x_1^2 + 2(b-a)\sin\theta\cos\theta\, x_1y_1 + (a\sin^2\theta + b\cos^2\theta)\,y_1^2 = 1$$

which is the form $a_1x_1^2 + 2h_1x_1y_1 + b_1y_1^2 = 1$ —①

From now on, we may regard a <u>rotated</u> ellipse or hyperbola has a general equation:



2.1 $$ax^2+2hxy+by^2 = 1$$

2.2 Invariants

In equation ① $\quad a_1+b_1 = a\cos^2\theta + b\sin^2\theta + a\sin^2\theta + b\cos^2\theta$
$\quad = a+b$

$a_1b_1 - h_1^2 = (a\cos^2\theta + b\sin^2\theta)(a\sin^2\theta + b\cos^2\theta) - (b-a)^2\sin^2\theta\cos^2\theta$
$\quad = ab - h^2$ (after simplification, and $h=0$)

The term $a+b$ and $ab-h^2$ are called invariants

2.3 Angle of rotation

Consider an equation $ax^2+2hxy+by^2=1$

we may want to transform the above eqt. to standard one.

Apply $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ to $ax^2+2hxy+by^2=1$

$\Rightarrow a(x_1\cos\theta - y_1\sin\theta)^2 + 2h(x_1\cos\theta - y_1\sin\theta)(x_1\sin\theta + y_1\cos\theta) + b(x_1\sin\theta + y_1\cos\theta)^2 = 1$

coefficient of x_1y_1 vanished

$\Rightarrow -2a\cos\theta\sin\theta + 2h(\cos^2\theta - \sin^2\theta) + 2b\sin\theta\cos\theta = 0$

$\Rightarrow \quad 2h\cos 2\theta = (a-b)\,2\sin\theta\cos\theta$

(*) $\Rightarrow \quad \boxed{\tan 2\theta = \dfrac{2h}{a-b}}$

from which we can find out the angle of rotation

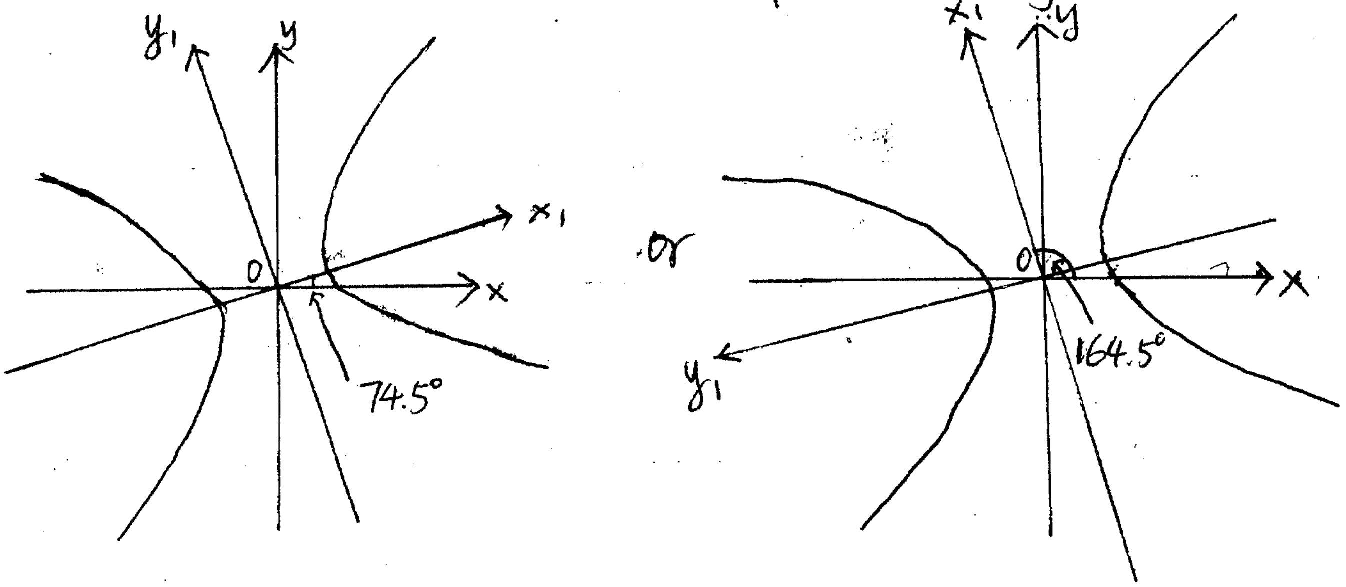
eg $\quad x^2 - 3xy - 4y^2 = 1$

$$\tan 2\theta = \frac{-3}{1-(-4)}$$

$$\tan 2\theta = -\frac{3}{5}$$

$$2\theta = 149^\circ \text{ or } 329^\circ$$
$$\theta = 74.5^\circ \text{ or } 164.5^\circ$$

2 different rotations is possible. The graphs as follows:



<u>The axes</u>

from (*) $\tan 2\theta = \frac{2h}{a-b}$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2h}{a-b}$$

$$\Rightarrow h(\tan\theta)^2 + (a-b)\tan\theta - h = 0$$

Two values of $\tan\theta$ can be found : m_1, m_2

Suppose the x_1-axis , y_1-axis are $y = m_1 x$, $y = m_2 x$ respestively referring to $x-y$ axes

Then the pair of principal axes is :

$$(m_1 x - y)(m_2 x - y) = 0$$

$$m_1 m_2 x - (m_1 + m_2) xy + y^2 = 0$$

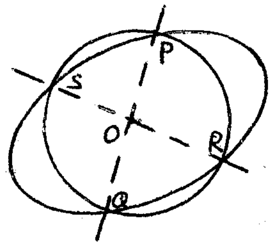
$$-\frac{h}{h} x^2 + \frac{a-b}{h} xy + y^2 = 0$$

$$\Rightarrow \boxed{hx^2 - (a-b)xy - hy^2 = 0}$$

2.4 Condition for ellipse or hyperbola

Suppose $ax^2 + 2hxy + by^2 = 1$

intersects with $x^2 + y^2 = r^2$ — a circle

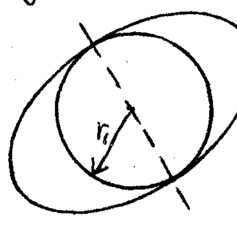


then $ax^2 + 2hxy + by^2 = (\frac{x}{r})^2 + (\frac{y}{r})^2$

$\Rightarrow$ ① $(a - \frac{1}{r^2})x^2 + 2hxy + (b - \frac{1}{r^2})y^2 = 0$

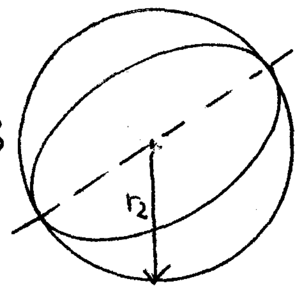
This is the equation of a pair of straight lines through origin.

If we change the radius r so that the 2 lines PQ, RS coincide (重疊) then the discriminant of equation ① $= 0$



ie $h^2 - (a - \frac{1}{r^2})(b - \frac{1}{r^2}) = 0$

(*) $\Rightarrow$ $\boxed{(h^2 - ab)r^4 + (a+b)r^2 - 1 = 0}$



From the above equation, we find 2 roots α, β

ie $r^2 = \alpha$ or β

If $ax^2 + 2hxy + by^2 = 1$ is an ellipse,

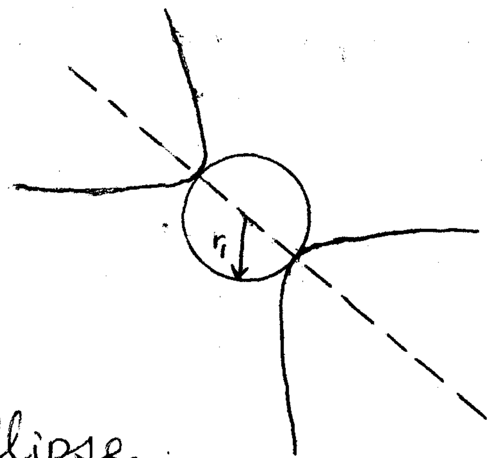
we can find 2: r_1, r_2

$\therefore$ $\alpha > 0$ $\beta > 0$

ie $\alpha\beta > 0$ and $\alpha + \beta > 0$

$\frac{-1}{h^2 - ab} > 0$ and $-\frac{a+b}{h^2 - ab} > 0$

$\Rightarrow$ $\boxed{ab - h^2 > 0}$ $\boxed{a + b > 0}$



This is the condition for ellipse.

If it is a hyperbola, we can find only one value of r_1

ie $\alpha\beta < 0$

$\Rightarrow$ $\boxed{ab - h^2 < 0}$ condition for hyperbola

note that r in the equation (*) gives the semi-major axis, semi-minor axis or semi-transverse axis.

example 1 $\quad 41x^2-24xy+34y^2=50$

$\Rightarrow \frac{41}{50}x^2+2\times\left(\frac{-6}{25}\right)xy+\frac{34}{50}y^2=1$

$ab-h^2=\frac{41\times 34}{2500}-\left(\frac{144}{2500}\right)>0$

$a+b=\frac{41+34}{50}>0$

$\therefore$ It is an ellipse.

example 2 $\quad x^2-3xy-4y^2=1$

$ab-h^2=1\times(-4)-\left(-\frac{3}{2}\right)^2<0$

$\therefore$ It is a hyperbola

example 3 $\quad -2x^2+2xy-3y^2=1$

$ab-h^2=(-2)\times(-3)-1>0$

$a+b=-2-3<0$

It has no locus (why?)

(note that the equation may be written as:

$(x-y)^2+x^2+2y^2+1=0$

LHS positive $\quad$ RHS $=0$. is it possible?)

The case $ab-h^2=0$ will be discussed later.

Conclusion: $\quad ax^2+2hxy+by^2=1$

Conditions	$ab-h^2>0$		$ab-h^2<0$
	$a+b>0$	$a+b<0$	
Locus name	ellipse	no locus	hyperbola
example	$41x^2-24xy+34y^2=50$	$-2x^2+2xy-3y^2=1$	$x^2-3xy-4y^2=1$

The asymptotes

Suppose $ax^2+2hxy+by^2=1$ is a hyperbola

$\Rightarrow \quad ab-h^2<0$

Suppose the curve intersect with $y=mx$

$\Rightarrow \quad ax^2+2hx(mx)+b(mx)^2=1$

$\Rightarrow \quad (a+2hm+bm^2)x^2-1=0$

This line is an asymptote if it touches the curve at $\pm\infty$

This is possible only if the coefficients of x^2 and x are zero

$\Rightarrow \quad a+2hm+bm^2=0$

-an quadratic equation in m gives m_1, m_2

If $y=m_1x$, $y=m_2x$ are the 2 asymptotes

then $(m_1x-y)(m_2x-y)=0$

$\Rightarrow \quad m_1m_2x^2-(m_1+m_2)xy+y^2=0$

$\Rightarrow \quad \frac{a}{b}x^2+\frac{2hxy}{b}+y^2=0$

$\Rightarrow \quad \boxed{ax^2+2hxy+by^2=0}$

note that the equation of asymptotes differ from the curve by a constant. (Please refer to 1.7)

2.5 When $ab - h^2 = 0$

$ab = h^2 \geqslant 0$

$a \geqslant 0,\ b \geqslant 0$ or $a \leqslant 0,\ b \leqslant 0$

when $a > 0,\ b > 0$

$ax^2 + 2hxy + by^2 = 1$

$\Rightarrow (\sqrt{a}x \pm \sqrt{b}y)^2 = 1$

$\Rightarrow (\sqrt{a}x \pm \sqrt{b}y)^2 - 1^2 = 0$

$\Rightarrow (\sqrt{a}x \pm \sqrt{b}y + 1)(\sqrt{a}x \pm \sqrt{b}y - 1) = 0$

which is a pair of straight parallel lines.
(not passing through origin)

When $a < 0,\ b < 0$

$ax^2 + 2hxy + by^2 = 1$

$\Rightarrow -(\sqrt{-a}x \pm \sqrt{-b}y)^2 = 1$

$\Rightarrow$ no locus

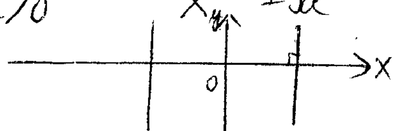
when $a \neq 0,\ b = 0 \Rightarrow h = 0$

$ax^2 + 2hxy + by^2 = 1$

$ax^2 = 1$

$a < 0$ no locus

$a > 0$ $x = \pm\frac{1}{\sqrt{a}}$ 2 lines // to y-axis



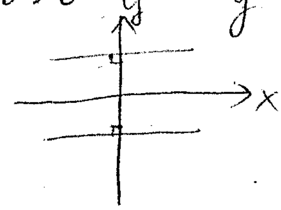
when $a = 0,\ b \neq 0 \Rightarrow h = 0$

$ax^2 + 2hxy + by^2 = 1$

$\Rightarrow by^2 = 1$

$b < 0$ no locus

$b > 0$ $y = \pm\frac{1}{\sqrt{b}}$



2 horizontal lines

When $a = b = 0$

$ab - h^2 = 0 \Rightarrow h = 0$

$ax^2 + 2hxy + by^2 = 1$

$\Rightarrow 0 = 1$

$\Rightarrow$ no locus.