Conic Section in General

Reference: Advanced Level Pure Mathematics by S.L. Green p.103-104

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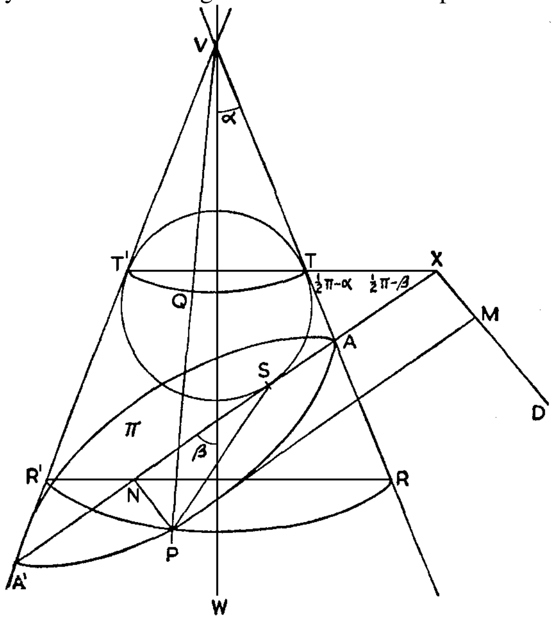

If a straight line *VA* (produced indefinitely far both ways) is rotated about a fixed line *VW*, the angle *WVA* being kept constant, the surface which it generates is a *right circular cone* in the figure. The point *V* is the vertex; *VW* is the axis; and the acute angle *WVA* (= α) is the *semi-vertical angle* of the cone. *VA* in any of its positions during the rotation is a *generator* of the surface.

Clearly, the section (切面) of the surface by a plane through *V* consists of two straight lines, each of them a generator (*VA* and *VA*'); the section by a plane at right angle to *VW* is a circle (with diameter *TT* '). It will be proved that the curve in which the surface is cut by a plane, not perpendicular to the axis and not containing *V*, is a parabola, an ellipse, or a hyperbola.

Consider the section by a plane π. Let the plane through *VW* perpendicular to π meet the surface in the two generators *VA*, *VA*', the points *A*, *A*' lying in the plane π. Through any point *P* on the curve of intersection draw *PN* at right angles to *AA*' and through *PN* draw a plane at right angle to *VW*. This plane, which cuts the cone in the circle, meets the generators *VA*, *VA*' in *R* and *R*', diametrically opposite points on the circle.

Let the circle inscribed in the triangle *VAA*' touch *AA*' at *S*, *VA* at *T*, and *VA*' at *T* '; then *AS* = *AT*, since they are tangents from *A* to the circle. If this circle is rotated about *VW*, it generates a sphere which touches the cone at all points of the circle generated in the rotation by *T* (and *T* '); let the plane of this circle meet π in the line *XD*, where *X* lies on *AA*'. The *XD* is at right angle to *A*'*A*.

If *VP* meets the circle generated by *T* in *Q*, *RT* = *PQ* = *PS* (tangents to a sphere). Also, if *PM* is perpendicular to *XD*, *PM* = *NX*.

Now $SP : PM = PQ : NX = RT : NX$.

The triangles *ANR*, *AXT* being similar,

$AR : AN = AT : AX = (AR + AT) : (AN + AX) = RT : NX = SP : PM$.

Thus $SP = ePM$,

where $e = \dfrac{AT}{AX} = \dfrac{\cos\beta}{\cos\alpha}$, the acute angle β being the inclination of the plane π to the axis of the cone.

It follows that the section of the cone is

(i) a parabola, if β = α, *i.e.* if the plane of section is parallel to the generator (*VA*') of the cone;

or (ii) an ellipse, if β > α, *i.e.* if the plane cuts the cone wholly on one side of the vertex;

or (iii) a hyperbola, if β < α, *i.e.* if the plane cuts the cone on both sides of the vertex.

In any case, the eccentricity of this section is equal to $\frac{\cos\beta}{\cos\alpha}$, S is a focus and XD the corresponding directrix.

Since the parabola, ellipse, and hyperbola may be obtained as plane sections of a cone they are known as *conic sections* or *conics*. As particular cases of conic sections we have the circle, where β is a right angle, and a pair of straight lines when the plane π passes through the vertex.