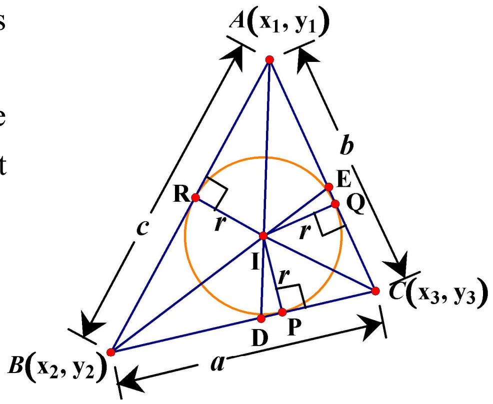
Coordinates of Incentre of a triangle

Created by Mr. Francis Hung on 20230530. Last upated: 05/08/2023



Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the coordinates of the vertices of ΔABC. $BC = a$, $CA = b$, $AB = c$. Let I be the incentre.

Let the radius of the inscribed circle be r. The inscribed circle touches ΔABC at P, Q and R. Join AI and produce it to cut BC at D. Denote the areas by S.

BC, CA and AB are tangents to the inscribed circle.

$IP \perp BC$, $IQ \perp AC$, $IR \perp AB$ (tangent $\perp$ radii)

$IP = IQ = IR = r$

$S_{\Delta IBC} + S_{\Delta ICA} + S_{\Delta IAB} = S_{\Delta ABC}$

$\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \sqrt{s(s-a)(s-b)(s-c)}$ Heron's formula, where $s = \frac{1}{2}(a+b+c)$

$sr = \sqrt{s(s-a)(s-b)(s-c)}$

$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$, where $s = \frac{1}{2}(a+b+c)$

Apply angle bisector theorem on ΔABC.

$$\frac{BD}{DC} = \frac{c}{b} \Rightarrow D = \left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c}\right)$$

$$CD = \frac{ab}{b+c}$$

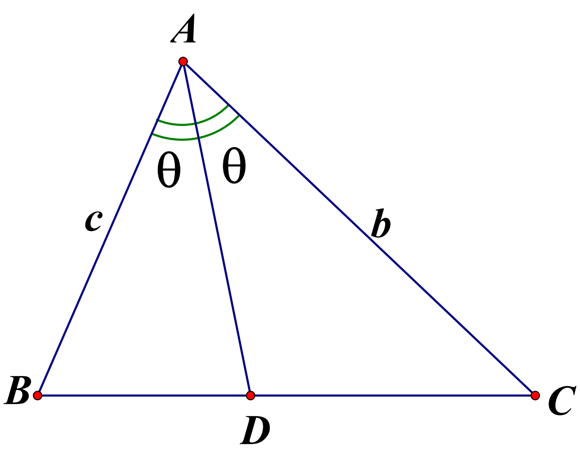
Join CI and apply angle bisector theorem on ΔACD.

$$\frac{AI}{ID} = \frac{b}{\frac{ab}{b+c}} \Rightarrow \frac{AI}{ID} = \frac{b+c}{a}$$

$$I = \left(\frac{ax_1 + (b+c)\cdot\frac{bx_2 + cx_3}{b+c}}{a+b+c}, \frac{ay_1 + (b+c)\cdot\frac{by_2 + cy_3}{b+c}}{a+b+c}\right)$$

$$= \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

Remark: angle bisector theorem



In the figure, $AC = b$, $AB = c$, AD is the angle bisector of $\angle A$, cutting BC at D. $\angle BAD = \angle CAD = \theta$.

Then $\frac{BD}{DC} = \frac{c}{b}$.

Let $\angle ADB = \alpha$, $\angle ADC = 180° - \alpha$ (adj. $\angle$s on st. line)

Apply sine rule on ΔABD and ΔACD.

$$\frac{BD}{\sin\theta} = \frac{c}{\sin\alpha} \cdots (1) \text{ and } \frac{DC}{\sin\theta} = \frac{b}{\sin(180° - \alpha)} \cdots (2)$$

Using the fact that $\sin(180° - \alpha) = \sin\alpha$, (1) ÷ (2):

$$\frac{BD}{DC} = \frac{c}{b}$$