Equation of tangent of a circle at $P(x_0, y_0)$ on the circle.

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Given a circle with equation C: $x^2 + y^2 + Dx + Ey + F = 0$.

The cerntre is $G = \left(-\frac{D}{2}, -\frac{E}{2}\right)$, radius is $r = \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{E}{2}\right)^2 - F}$ (assume that $D^2 + E^2 - 4F \geq 0$.)

Let $P(x_0, y_0)$ be any point on the circle. i.e. $x_0^2 + y_0^2 + Dx_0 + Ey_0 + F = 0$ …… (*)

Then the equation of tangent is: $x_0 x + y_0 y + \frac{D}{2}(x + x_0) + \frac{E}{2}(y + y_0) + F = 0$.

Proof: Slope of $GP = \dfrac{y_0 - \left(-\frac{E}{2}\right)}{x_0 - \left(-\frac{D}{2}\right)} = \dfrac{y_0 + \frac{E}{2}}{x_0 + \frac{D}{2}}$

Slope of tangent through $P = \dfrac{-1}{\text{slope of } GP} = -\dfrac{x_0 + \frac{D}{2}}{y_0 + \frac{E}{2}}$

By point-slope form, the equation of tangent through P is $y - y_0 = -\dfrac{x_0 + \frac{D}{2}}{y_0 + \frac{E}{2}}(x - x_0)$.

$$\left(y_0 + \frac{E}{2}\right)(y - y_0) = -\left(x_0 + \frac{D}{2}\right)(x - x_0)$$
$$\left(x_0 + \frac{D}{2}\right)(x - x_0) + \left(y_0 + \frac{E}{2}\right)(y - y_0) = 0$$
$$\left(x_0 + \frac{D}{2}\right)x - x_0\left(x_0 + \frac{D}{2}\right) + \left(y_0 + \frac{E}{2}\right)y - \left(y_0 + \frac{E}{2}\right)y_0 = 0$$
$$x_0 x + y_0 y + \frac{D}{2}(x + x_0) + \frac{E}{2}(y + y_0) + F - \left(x_0^2 + y_0^2 + Dx_0 + Ey_0 + F\right) = 0$$
$$x_0 x + y_0 y + \frac{D}{2}(x + x_0) + \frac{E}{2}(y + y_0) + F = 0 \quad \text{(By (*))}$$

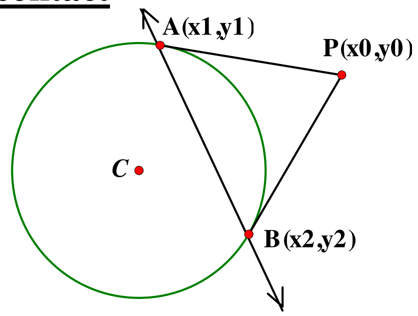
The proof is completed.

The chord of contact

If $P(x_0, y_0)$ lies ooutside the circle C, then the equation

$$x_0x + y_0y + \frac{D}{2}(x + x_0) + \frac{E}{2}(y + y_0) + F = 0$$

becomes the contact of contact.



Two external tangents can be drawn through the external point $P(x_0, y_0)$.

Suppose the coordinates of the points of contact are $A(x_1, y_1)$ and $B(x_2, y_2)$.

Then, by the above result, the equations of PA and PB are:

PA: $x_1x + y_1y + \frac{D}{2}(x + x_1) + \frac{E}{2}(y + y_1) + F = 0$ …… (1)

PB: $x_2x + y_2y + \frac{D}{2}(x + x_2) + \frac{E}{2}(y + y_2) + F = 0$ …… (2)

$\because$ PA and PB pass through P

$\therefore$ $x_1x_0 + y_1y_0 + \frac{D}{2}(x_0 + x_1) + \frac{E}{2}(y_0 + y_1) + F = 0$ …… (3)

$x_2x_0 + y_2y_0 + \frac{D}{2}(x_0 + x_2) + \frac{E}{2}(y_0 + y_2) + F = 0$ …… (4)

Consider the equation: $x_0x + y_0y + \frac{D}{2}(x + x_0) + \frac{E}{2}(y + y_0) + F = 0$

Clearly it is a linear equation, so it is a straight line.

Put $x = x_1$, and $y = y_1$, it becomes equation (3). $\therefore$ This straight line passes through (x_1, y_1).

Put $x = x_2$, and $y = y_2$, it becomes equation (4). $\therefore$ This straight line passes through (x_2, y_2).

$\therefore$ The equation of the chord of contact is: $x_0x + y_0y + \frac{D}{2}(x + x_0) + \frac{E}{2}(y + y_0) + F = 0$.

Radical axis of two circles

Given the following two different circles:

C_1: $x^2 + y^2 + D_1x + E_1y + F_1 = 0$

C_2: $x^2 + y^2 + D_2x + E_2y + F_2 = 0$

Consider $C_1 - C_2$: $(D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2 = 0$.

The equation is linear in x and linear is y, so it is a straight line. This line is called L, the **radical axis**.

(1) The radical axis passes through a line which is perpendicular to the line joining the centres of C_1 and C_2.

Proof: L: $(D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2 = 0$

Centres $G_1\left(-\frac{D_1}{2}, -\frac{E_1}{2}\right)$, $G_2\left(-\frac{D_2}{2}, -\frac{E_2}{2}\right)$

$$\text{Product of slopes} = -\frac{D_1 - D_2}{E_1 - E_2} \cdot \frac{-\frac{E_2}{2} + \frac{E_1}{2}}{-\frac{D_2}{2} + \frac{D_1}{2}}$$

$$= -\frac{D_1 - D_2}{E_1 - E_2} \cdot \frac{E_1 - E_2}{D_1 - D_2} = -1$$

∴ They are perpendicular.

(2) If C_1 and C_2 intersect at $P(x_3, y_3)$ and $Q(x_4, y_4)$, then PQ is the radical axis.

$x_3^2 + y_3^2 + D_1x_3 + E_1y_3 + F_1 = 0$ …… (5)

$x_4^2 + y_4^2 + D_1x_4 + E_1y_4 + F_1 = 0$ …… (6)

$x_3^2 + y_3^2 + D_2x_3 + E_2y_3 + F_2 = 0$ …… (7)

$x_4^2 + y_4^2 + D_2x_4 + E_2y_4 + F_2 = 0$ …… (8)

(5) – (7): $(D_1 - D_2)x_3 + (E_1 - E_2)y_3 + F_1 - F_2 = 0$ …… (9)

(6) – (8): $(D_1 - D_2)x_4 + (E_1 - E_2)y_4 + F_1 - F_2 = 0$ …… (10)

By (9), the radical axis L passes through (x_3, y_3).

By (10), the radical axis L passes through (x_4, y_4).

The proof is completed.