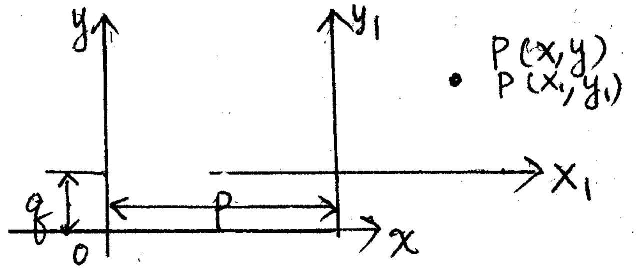
3 General Conics.



Suppose we translate the co-ordinate axes to (p, q)
then $\quad x = x_1 + p \qquad y = y_1 + q$

$$ax^2 + 2hxy + by^2 = 1$$
$$\Rightarrow a(x_1+p)^2 + 2h(x_1+p)(y_1+q) + b(y_1+q)^2 = 1$$
$$\Rightarrow ax_1^2 + 2hx_1y_1 + by_1^2 + 2(ap+hq)x + 2(hp+bq)y + ap^2 + 2hpq + bq^2 - 1 = 0$$

$\therefore$ In general, a conics has a general equation:

3.1 $\quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

3.2 Conversely, given $\quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
transform $\quad x = x_1 + p, \; y = y_1 + q$

$$\Rightarrow a(x_1+p)^2 + 2h(x_1+p)(y_1+q) + b(y_1+q)^2 + 2g(x_1+p) + 2f(y_1+q) + c = 0$$

(*) $$\Rightarrow ax_1^2 + 2hx_1y_1 + by_1^2 + ap^2 + 2hpq + bq^2 + 2gp + 2fq + c + 2(ap+hq+g)x_1 + 2(hp+bq+f)y_1 = 0$$

make the coefficient of $x_1 = 0$, coefficient of $y_1 = 0$

ie $\quad ap + hq + g = 0 \quad$ — ①
$\quad\;\; hp + bq + f = 0 \quad$ — ②

$$\boxed{\Rightarrow \begin{cases} p = \dfrac{fh - bg}{ab - h^2} \\ q = \dfrac{gh - af}{ab - h^2} \end{cases}} \qquad \text{provided that } ab - h^2 \neq 0$$

This is called the centre of conics.

3.3 let C_1 = constant term of (*) in 3.2

$$
\begin{aligned}
&= ap^2+2hpq+bq^2+2gp+2fq+c \\
&= ap^2+hpq+gp \\
&\quad + hpq+bq^2+fq \\
&\quad + gp+fq+c \\
&= p(ap+hq+g)+q(hp+bq+f)+gp+fq+c \\
&= \underset{\uparrow \text{by } ①}{0} + \underset{\uparrow \text{by } ②}{0} + gp+fq+c \\
&= g\cdot\frac{fh-bg}{ab-h^2}+f\,\frac{gh-af}{ab-h^2}+c \\
&= \frac{fgh-bg^2+fgh-af^2+c(ab-h^2)}{ab-h^2}
\end{aligned}
$$

$$\boxed{C_1=\frac{\Delta}{ab-h^2}} \quad \text{where } \Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Example $3x^2+2xy+3y^2+6x+10y-9=0$

$ab-h^2=9-1=8$

$a+b=6$

$$\Delta=\begin{vmatrix} 3 & 1 & 3 \\ 1 & 3 & 5 \\ 3 & 5 & -9 \end{vmatrix}=-81+15+15-27+9-75=-144$$

$(a+b)\Delta=6\times(-144)<0$, $a+b>0$

$\therefore$ It is an ellipse

centre $=\left(\frac{5-9}{8},\frac{3-15}{8}\right)=\left(-\frac{1}{2},-\frac{3}{2}\right)$

Translated equation: $3x_1^2+2x_1y_1+3y_1^2-\frac{144}{8}=0$

$$\Rightarrow \frac{x_1^2}{6}+\frac{2x_1y_1}{18}+\frac{y_1^2}{6}=1$$

Let $f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

If $ab - h^2 \neq 0$ $f(x,y) = 0$ can be transformed to

$ax_1^2 + 2hx_1y_1 + by_1^2 + c_1 = 0$, where $c_1 = \frac{\Delta}{ab-h^2}$

which is $\begin{cases} \text{a pair of Straight Line if } c_1 = 0 \\ \text{a central conics if } c_1 \neq 0 \end{cases}$

In the case $c_1 \neq 0$, $f(x_1, y_1) = -c_1$ can be written as

$$\frac{a}{-c_1} x_1^2 + \frac{2h}{-c_1} x_1 y_1 + \frac{b}{-c_1} y_1^2 = 1$$

If $\frac{a}{-c_1}\frac{b}{-c_1} - \left(\frac{h}{-c_1}\right)^2 < 0$ then it is a hyperbola.

$\Leftrightarrow \quad ab - h^2 < 0$

If $\frac{a}{-c_1}\frac{b}{-c_1} - \left(\frac{h}{-c_1}\right)^2 > 0$ and $\frac{a}{-c_1} + \frac{b}{-c_1} > 0$ then it is an ellipse

(refer to section 2.4)

$\Leftrightarrow \quad ab - h^2 > 0$ and $\frac{a+b}{-c_1} > 0$

$\Leftrightarrow \quad ab - h^2 > 0$ and $\frac{a+b}{\frac{-\Delta}{ab-h^2}} > 0$

$\Leftrightarrow \quad ab - h^2 > 0$ and $\frac{a+b}{-\Delta} > 0$

$\Leftrightarrow \quad ab - h^2 > 0$ and $\frac{a+b}{\Delta} < 0$

$\Leftrightarrow \quad ab - h^2 > 0$ and $(a+b)\Delta < 0$

$\therefore$

Condition for ellipse	Condition for hyperbola
$ab-h^2>0$ and $(a+b)\Delta<0$	$ab-h^2<0$

3.4 Suppose $ax^2+2hxy+by^2+2gx+2fy+c=0$ is a hyperbola
ie $ab-h^2<0$
we may want to find out the equation of asymptotes
Note that the asymptotes has the same centre as the curve.

(✱) Consider $ax^2+2hxy+by^2+2gx+2fy+c''=0$ a second degree eqt.
which has the same centre as the given conics
suppose this equation satisfies the condition

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c'' \end{vmatrix} = 0 \quad \text{and} \quad ab-h^2<0$$

then (✱) represents a pair of straight lines.

$$\Rightarrow \quad abc''+2fgh-af^2-bg^2-c''h^2=0$$

$$\Rightarrow \quad (ab-h^2)c''+2fgh-af^2-bg^2 = 0$$

$$\Rightarrow \quad (ab-h^2)c+2fgh-af^2-bg^2 = (ab-h^2)c-(ab-h^2)c''$$

$$\Rightarrow \quad \frac{\Delta}{ab-h^2} = c-c''$$

$$\Rightarrow \quad \boxed{c'' = c-\frac{\Delta}{ab-h^2}}$$

∴ The asymptotes are given by :

$$\boxed{ax^2+2hxy+by^2+2gx+2fy+c-\frac{\Delta}{ab-h^2}=0}.$$

4 <u>Rotated Parabola</u>

Given $y^2 = 4ax$

Transform the equation by: $x = x_1\cos\theta - y_1\sin\theta + p$
$y = x_1\sin\theta + y_1\cos\theta + q$

(rotation then by a translation)

$\Rightarrow (x_1\sin\theta + y_1\cos\theta + q)^2 = 4a(x_1\cos\theta - y_1\sin\theta + p)$

which is a form $a_1x_1^2 + 2h_1x_1y_1 + b_1y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 = 0$

It can be shown that $x^2 = 4ay$ is also transformed to a second degree general equation. (exercise)

note that $\boxed{a_1b_1 - h_1^2 = 0}$

proof: $a_1b_1 - h_1^2 = \sin^2\theta\cos^2\theta - (\sin\theta\cos\theta)^2$
$= 0$ * QED

4 Rotated Parabola

18

let $f(x,y) = ax^2+2hxy+by^2+2gx+2fy+c=0$

Suppose $\Delta \neq 0$ and $ab-h^2=0$

then $h^2 = ab$

we may assume that $h \neq 0$ (otherwise it can be done by completing the square)

$\therefore \quad h^2 = ab > 0$

WLOG we may also assume that $a>0$, $b>0$

we use the rotation: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

then $f(x,y)=0$ is transformed to:

$$a_1x_1^2+2h_1x_1y_1+b_1y_1^2+2g_1x_1+2f_1y_1+c=0$$

for a suitable choice of θ, we can make $h_1=0$

in this case $\tan 2\theta = \dfrac{2h}{a-b}$ (please read section 2.3)

$$\Rightarrow h(\tan\theta)^2-(a-b)\tan\theta-h=0$$

$$\Rightarrow \tan\theta = \frac{(b-a)\pm\sqrt{a^2+b^2+2h^2}}{2h}, \quad h\neq 0$$

$$\Rightarrow \tan\theta = \frac{(b-a)\pm(a+b)}{2h}, \quad h^2=ab$$

$$= \frac{b}{h} \text{ or } -\frac{a}{h}$$

we choose the angle of rotation θ so that $0°<\theta<90°$

$\Rightarrow \tan\theta > 0$

Case 1 $\quad h>0$

$\Rightarrow h = \sqrt{ab}$

$\tan\theta = \dfrac{b}{\sqrt{ab}} = \sqrt{\dfrac{b}{a}}$ (reject $\tan\theta = -\dfrac{a}{h}$)

After a little manipulation,

$$a_1 = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta$$

$$b_1 = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta$$

$$\Rightarrow \begin{cases} a_1 = a\cos^2\theta + 2\sqrt{ab}\sin\theta\cos\theta + b\sin^2\theta \\ b_1 = a\sin^2\theta - 2\sqrt{ab}\sin\theta\cos\theta + b\cos^2\theta \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = (\sqrt{a}\cos\theta + \sqrt{b}\sin\theta)^2 \\ b_1 = (\sqrt{a}\sin\theta - \sqrt{b}\cos\theta)^2 \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = \left(\sqrt{a}\frac{\sqrt{a}}{\sqrt{a+b}} + \sqrt{b}\frac{\sqrt{b}}{\sqrt{a+b}}\right)^2 \\ b_1 = \left(\sqrt{a}\frac{\sqrt{b}}{\sqrt{a+b}} - \sqrt{b}\frac{\sqrt{a}}{\sqrt{a+b}}\right)^2 \end{cases}, \quad \because \tan\theta = \sqrt{\frac{b}{a}}$$

$$\Rightarrow \begin{cases} a_1 = a+b \\ b_1 = 0 \end{cases}$$

$$\begin{cases} g_1 = g\cos\theta + f\sin\theta \\ f_1 = f\cos\theta - g\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} g_1 = g\frac{\sqrt{a}}{\sqrt{a+b}} + f\frac{\sqrt{b}}{\sqrt{a+b}} \\ f_1 = f\frac{\sqrt{a}}{\sqrt{a+b}} - g\frac{\sqrt{b}}{\sqrt{a+b}} \end{cases}$$

$$\Rightarrow \begin{cases} g_1 = \frac{\sqrt{a}g + \sqrt{b}f}{\sqrt{a+b}} \\ f_1 = \frac{\sqrt{a}f - \sqrt{b}g}{\sqrt{a+b}} \end{cases}$$

Note that $\Delta \neq 0$

$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

$\Rightarrow (fh - bg)^2 - (h^2 - ab)(f^2 - bc) \neq 0$

$\Rightarrow fh - bg \neq 0$

$\Rightarrow f\sqrt{ab} - bg \neq 0$

$\Rightarrow \sqrt{a}f - \sqrt{b}g \neq 0$

Therefore the transformed equation is:

$$(a+b)x_1^2 + 2g_1x_1 + 2f_1y_1 + c = 0$$

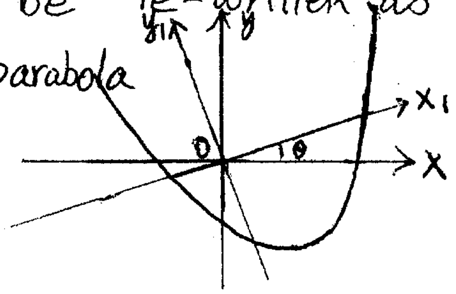
which may be re-written as:

$$y_1 = Ax_1^2 + Bx_1 + C$$

$$A = \frac{a+b}{-2f_1} \qquad C = \frac{c}{-2f_1}$$

$$B = \frac{g_1}{-f_1}$$

a rotated parabola



case 2 $\quad h<0$

$$\Rightarrow h=-\sqrt{ab}$$

$$\tan\theta=\frac{a}{\sqrt{ab}}=\sqrt{\frac{a}{b}} \qquad (\text{reject } \tan\theta=\frac{b}{h})$$

$$a_1=a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta$$

$$b_1=a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta$$

$$\Rightarrow\begin{cases}a_1=a\cos^2\theta-2\sqrt{ab}\sin\theta\cos\theta+b\sin^2\theta\\ b_1=a\sin^2\theta+2\sqrt{ab}\sin\theta\cos\theta+b\cos^2\theta\end{cases}$$

$$\Rightarrow\begin{cases}a_1=(\sqrt{a}\cos\theta-\sqrt{b}\sin\theta)^2\\ b_1=(\sqrt{a}\sin\theta+\sqrt{b}\cos\theta)^2\end{cases}$$

$$\Rightarrow\begin{cases}a_1=\left(\sqrt{a}\frac{\sqrt{b}}{\sqrt{a+b}}-\sqrt{b}\frac{\sqrt{a}}{\sqrt{a+b}}\right)^2=0\\ b_1=\left(\sqrt{a}\frac{\sqrt{a}}{\sqrt{a+b}}+\sqrt{b}\frac{\sqrt{b}}{\sqrt{a+b}}\right)^2=a+b\end{cases}$$

$$\begin{cases}g_1=g\cos\theta+f\sin\theta\\ f_1=f\cos\theta-g\sin\theta\end{cases}$$

$$\Rightarrow\begin{cases}g_1=g\frac{\sqrt{b}}{\sqrt{a+b}}+f\frac{\sqrt{a}}{\sqrt{a+b}}=\frac{\sqrt{b}g+\sqrt{a}f}{\sqrt{a+b}}\\ f_1=f\frac{\sqrt{b}}{\sqrt{a+b}}-g\frac{\sqrt{a}}{\sqrt{a+b}}=\frac{f\sqrt{b}-g\sqrt{a}}{\sqrt{a+b}}\end{cases}$$

Note that $\Delta\neq 0$

$$\Rightarrow abc+2fgh-af^2-bg^2-ch^2\neq 0$$

$$\Rightarrow (fh-bg)^2-(h^2-ab)(f^2-bc)\neq 0$$

$$\Rightarrow fh-bg\neq 0$$

$$\Rightarrow -\sqrt{ab}f-bg\neq 0$$

$$\Rightarrow \sqrt{a}f+\sqrt{b}g\neq 0 \Rightarrow g_1\neq 0$$

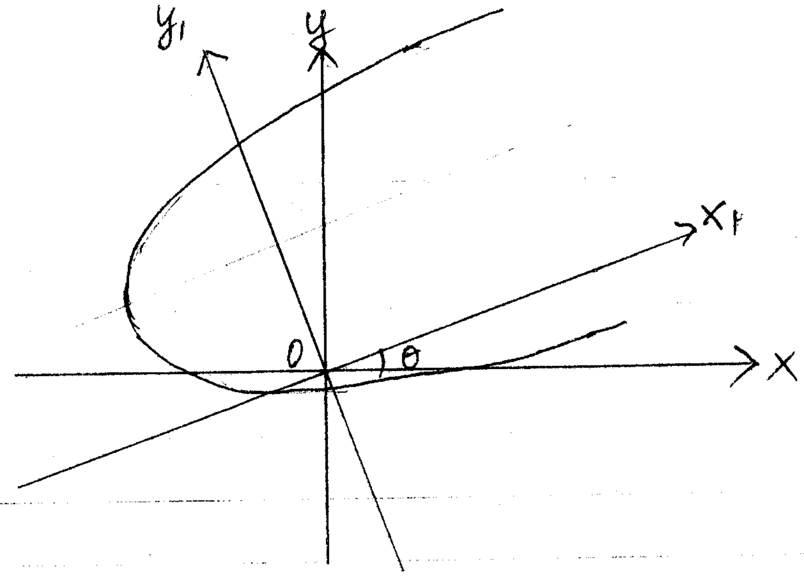
Therefore the transformed equation is:

$$(a+b)y_1^2+2g_1x+2f_1y+c=0$$

which may be re-written as: $x_1=Ay_1^2+By_1+C$

$$A=\frac{a+b}{-2g_1},\ B=\frac{f_1}{g_1},\ C=\frac{c}{-2g_1}$$

A rotated parabola:



Example: Consider $16x^2-24xy+9y^2-112x-166y+721=0$.

$a=16 \quad h=-12 \quad b=9 \quad g=-56 \quad f=-83 \quad c=721$

$ab-h^2=16\times9-12^2=0$

$$\Delta=\begin{vmatrix}16 & -12 & -56\\ -12 & 9 & -83\\ -56 & -83 & 721\end{vmatrix}=-26896\neq0$$

$\therefore$ It is a parabola ; $h<0$, use the result in case 2

$\tan\theta=\frac{b}{h}$ or $-\frac{a}{h}=\frac{9}{-12}$ or $-\frac{16}{-12}=-\frac{3}{4}$ or $\frac{4}{3}$.

$\because \tan\theta>0 \quad \therefore \tan\theta=\frac{4}{3}$ only $\quad \theta=53.13^\circ$

The rotated equation is: $(a+b)y_1^2+2g_1x_1+2f_1y_1+c=0$

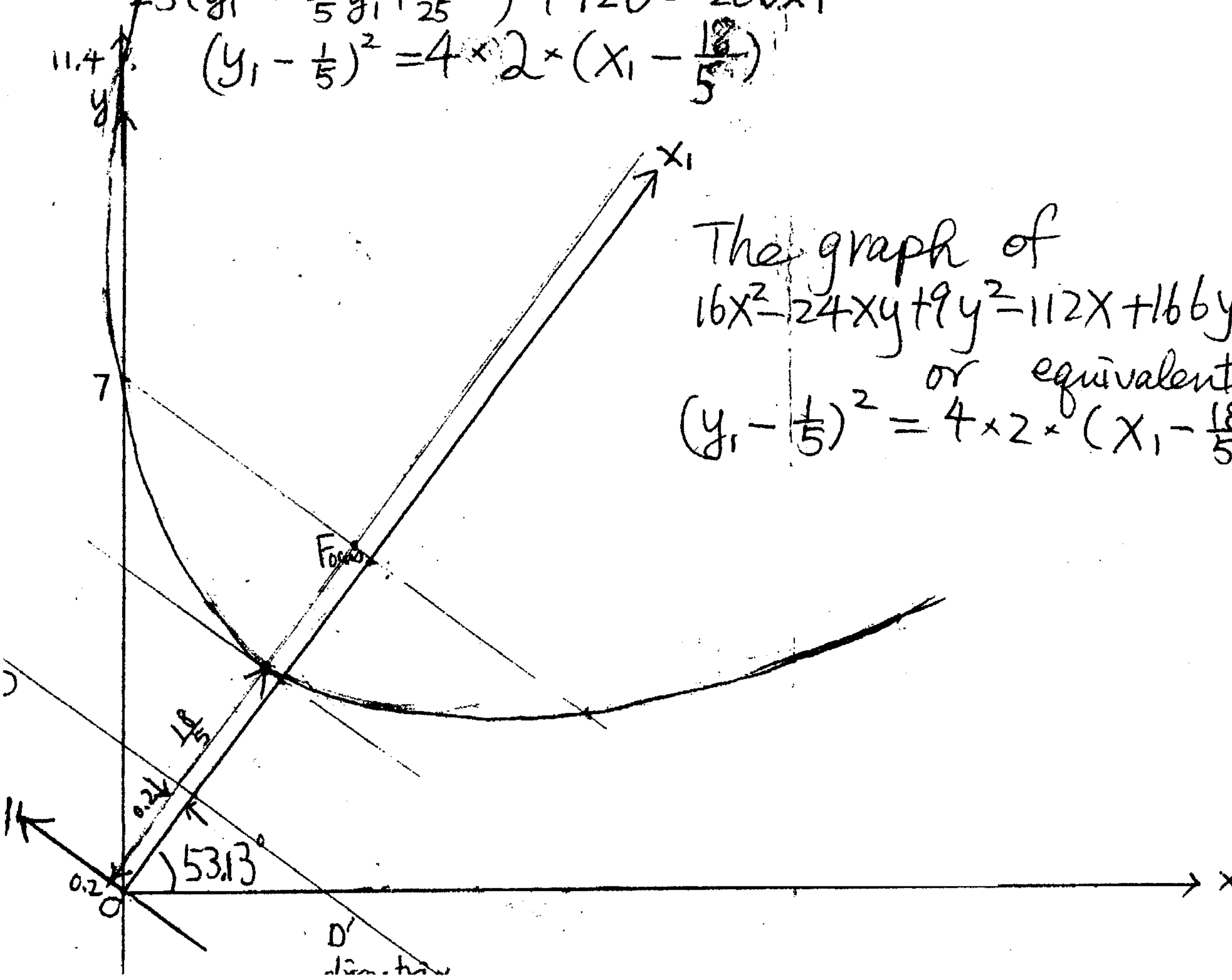
$$g_1=\frac{\sqrt{b}g+\sqrt{a}f}{\sqrt{a+b}}=\frac{3\times(-56)-4\times83}{\sqrt{25}}=-100$$

$$f_1=\frac{\sqrt{b}f-\sqrt{a}g}{\sqrt{a+b}}=-\frac{3\times83+4\times56}{\sqrt{25}}=-5$$

$$\Rightarrow 25y_1^2-200x_1-10y_1+721=0$$

$$25\left(y_1^2-\tfrac{2}{5}y_1+\tfrac{1}{25}\right)+720=200x_1$$

$$\left(y_1-\tfrac{1}{5}\right)^2=4\times2\times\left(x_1-\tfrac{18}{5}\right)$$



The graph of
$16x^2-24xy+9y^2-112x+166y+721=0$
or equivalently
$\left(y_1-\frac{1}{5}\right)^2=4\times2\times\left(x_1-\frac{18}{5}\right)$